1. Consider the following linear programming problem:

   - Minimize $\sum_{j=1}^{n} c_j x_j$ subject to:
   - $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, $i \in \{1, 2, \ldots m\}$,
   - $x_j \geq 0$.

   Write down the dual program and the dual of the dual. Show that the **dual of the dual is the primal** itself.

2. Given a graph $G(V, E)$, recall that

   1. A **vertex cover** in G is a subset C of V such that, for each edge $uv \in E$, $u \in C$ or $v \in C$ (or both). The **Minimum Vertex Cover Problem** (MVC) is to find a vertex cover of minimum size in G.
   2. An **independent set** in G is a subset I of V such that, for each $u, v \in I$, $uv \notin E$. The **Maximum Independent Set Problem** (MIS) is to find an independent set of maximum size in G.
   3. A **matching** in G is a subset M of E, such that if $uv \in M$, then for any $x \in V$, $u \neq x \neq v$, $ux, vx \notin M$. (That is, if $e \in M$, then no edge adjacent to e must be in M.). The **Maximum Matching Problem** (MM) is to find a matching of maximum size in G.
   4. An **edge cover** in G is a subset F of E, such that for each vertex $u \in V$, there exists $x \in V$ such that $ux \in F$. (That is, every vertex in G is "covered by" some edge in F). The **Minimum Edge Cover Problem** (MEC) is to find an edge cover of minimum size in G. (**Note:** A graph may not have any edge cover — why? Thus, this problem makes sense only for those graphs for which at least one edge cover exists. What is the requirement for a graph to have an edge cover?)

   Write down the primal and dual linear programming formulations for the fractional versions of each of the problems above. You will find the some problems are duals of some other problems. Find the duality relationship between the above problems.

3. The *Minimum Hitting Set Problem* (MHS) is defined as follows. Given a) a non-empty set U, b) a positive integer m and c) a collection of non-empty subsets $S_1, S_2, \ldots S_m$ of U. The problem is to find a subset H of U of **minimum size** such that $H \cap S_i \neq \emptyset$ for all $1 \leq i \leq m$. Write down a linear programming formulation for the fractional version of the problem. Write down the **dual** formulation.

4. The *Minimum Set Cover Problem* (MSC) takes as instance a) non-empty set U, b) a positive integer m and c) a collection of non-empty subsets $S_1, S_2, \ldots S_m$ of U. The problem is to pick minimum number of sets among $S_1, S_2, \ldots S_m$ such that the union of the selected sets is U. Write down an LP formulation for the fractional version of the problem. Write down the **dual** formulation.

5. Consider the following *fractional* variant of the KNAPSACK problem. There are n items in a shop. Item i has price q_i **per kilogram**. Assume that the items are listed in the descending order of price/kg. — that is, $q_1 \geq q_2 \geq \ldots \geq q_n$. Also given is a weight constraint W. You are allowed to pick any amount of any item, but the total weight of your pick is limited by W. (Note that here we assume that each item is amorphous, like sugar, and fractional amounts can be taken).

   1. Write down an LP formulation for the above problem and write down the dual.
   2. What is the value of the optimal solution for the dual? What can you conclude from this about the primal?
   3. Suppose there is an additional constraint in the above problem, that only b_i kilograms of item i is available in the shop. (Thus, you can pick at most b_i kilograms of item i). Add this constraint to your LP formulation and write down the dual.