Problems on graph theory with solutions

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Covering Problems

1. Let G be a graph such that the number of cycles in G of length at most g is at most $n/2$. Let the cardinality of the minimum vertex cover $\beta(G) = k$. Then show that there exists a subgraph G' of G with $\chi(G') \geq \frac{n}{2(n-k)}$ and girth $> g$.

   Solution: We delete one vertex from each cycle of G whose length is $\leq g$ to obtain graph G'. Hence G' contains $\geq n/2$ vertices. Clearly G' has girth $> g$. We have $\alpha(G') \leq \alpha(G) = n - \beta(G) = n - k$ and $\chi(G')\alpha(G') \geq |V(G')|$, hence the result.

2. Let A be an $n \times n$ matrix where the entry at the (i, j)th position a_{ij} is either 1 or 0 for $1 \leq i, j \leq n$. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column. Define the permanant of A, $\text{per}(A)$ as follows:

$$\text{per}(A) = \sum_{j=1}^{n} a_{1j}.\text{per}(A_{1j})$$

   Let G_A denote the bipartite graph with one part being $\{u_1, \ldots, u_n\}$ and the other part being $\{v_1, \ldots, v_n\}$ and u_i is adjacent to v_j if and only if $a_{ij} = 1$ in the matrix A. Show that $\text{per}(A) =$ the total number of perfect matchings in G_A.

   Solution: It can be observed that $\text{per}(A)$ is the sum of $n!$ terms where each term is the product of n elements. Every term is the product of elements taken from each of the n rows and columns. Hence, a perfect matching in G_A shows up in A as n 1's placed suitably, one in each row and one in each column. Hence the product of these elements is a term of $\text{per}(A)$. Therefore $\text{per}(A)$ counts the number of perfect matching of G_A.

3. Consider the matrix A defined in the previous question. Show that $\text{per}(A) = 0$ if and only if A contains an $s \times t$ zero submatrix such that $s + t = n + 1$.

   Solution: $\Rightarrow$: We know $perm(A) = 0$ *iff* there exists no perfect matching in G_A. From Hall's theorem, there exists a set $S \subseteq \{u_1, \ldots, u_n\}$ such that $|S| = s > N(S)$. Hence $N(S)$ contains at most $s - 1$ vertices. $V \setminus N(S)$ contains at least $n - s + 1 = t$ vertices. There is no edge from S to $V \setminus N(S)$. Hence there exits a sub-matrix with the required property.

   $\Leftarrow$: Let M be the $s \times t$ zero sub-matrix. Let $S = \{u_i : \ a_{ij} \in M\}$. Then $|S| = s$ and $|N(S)| \leq n - t = s - 1$, hence there is no perfect matching.

4. Consider a complete r-partite graph G having even number of vertices and with parts $A_1, A_2, \ldots, A_r$. Assume that $|A_1| \geq |A_2| \geq \cdots \geq |A_r| \geq 1$. Show that there exists a perfect matching in G if and only

if $|A_1| \leq \sum_{i=2}^{r} |A_i|$, using Tutte's theorem.

Solution: $\Rightarrow$: Suppose G has a perfect matching and $|A_1| > \sum_{i=2}^{r} |A_i|$. Then if $G - S = A_1$, S is a 'bad' set.
$\Leftarrow$: Suppose G does not have a perfect matching. Then there exists a 'bad' set S such that $q(G-S) > |S|$. If $G - S \neq A_k$ for some k, then $G - S$ is connected and the number of odd components of $G - S$ is either 0 or 1 which is $\leq |S|$ and hence S cannot be a bad set. Therefore $G - S$ has to be some A_k. $q(G - S)$ is maximum when $G - S = A_1$ but then $|A_1| \leq |S|$, hence there is no 'bad' set.

5. Show that in a bipartite graph a minimum vertex cover is a "barrier". (A set $S \subseteq V$ is a barrier if there exists a matching M such that the number of unmatched vertices with respect to M equals $q(G - S) - |S|$, where $q(G - S)$ is the number of odd components of $G - S$.)

   Solution: Let S be a minimum vertex cover. Consider a maximum matching M of the bipartite graph G (thus $|M| = |S|$). Then there is no matched edge in S. Hence, the number of matched vertices in $V - S$ is $|S|$. We have $|V - S| = q(G - S)$ and hence the result.

6. Let G be a bipartite graph with parts X and Y where $|X| = |Y|$. Let $d = \max(|S| - |N(S)| : S \subseteq X)$. Show that $\alpha'(G) = |X| - d$.

   Solution: $d \geq 0$ (consider $S = X$; if $N(S) \neq Y$ then $d > 0$, if $N(S) = Y$ then $d \geq 0$). Add d many vertices and connect each of them to every vertex of X. The resulting graph satisfies Hall's condition. Now construct a perfect matching for the graph and then delete the new vertices. Thus $\alpha'(G) = |X| - d$.

Connectivity

1. What is the vertex connectivity κ of $K_{m,n}$, the complete bipartite graph with m and n vertices on the two parts. Explain your answer.

   Solution: Let $m \leq n$. Then removing all m vertices of the smaller part disconnects the graph. Suppose $\kappa(K_{m,n}) \leq m - 1$. If a set of κ vertices intersects both parts, then its deletion cannot disconnect the graph. Upon deleting κ vertices of one part, the graph remains connected, hence $\kappa(K_{m,n}) = m$.

2. Let G be a simple graph of diameter two. Show that the edge connectivity of G is equal to its minimum degree, i.e. $\lambda = \delta(G)$.

   Solution: Suppose $\lambda \leq \delta - 1$. Let X, Y be two disjoint subsets of $V(G)$ such that the edges between them form a cut. If every vertex of X contributes an edge to the cut, then $|X| \leq \delta - 1$. Then every vertex of X has degree $\leq \delta - 1$, a contradiction. Hence X contains a vertex that does not contribute to the cut and so does Y. This implies the distance between these two vertices is at least 3, a contradiction to the diameter of the graph.

3. (a) Show that if G is simple and the minimum degree $\delta(G) \geq n - 2$, (n being the number of vertices in G) then the vertex connectivity $\kappa(G) = \delta(G)$.

   (b) For each $n \geq 4$, find a simple graph with $\delta(G) = n - 3$ and $\kappa(G) < \delta(G)$.

   Solution:

(a) If $\delta = n-1$, then G is a complete graph and the result follows. Let $\delta = n-2$. If $\kappa(G) = \delta(G)-1$, then there are $n-3$ vertices in an A, B separator of G. Then either A or B contains a single vertex and its degree is at most $n-3 = \delta - 1$, a contradiction.

(a) Let $V(G) = A_1 \cup B \cup A_2$ where $A_i \simeq K_2$ and $B \simeq K_{n-4}$ and every vertex of A_i is adjacent to every vertex of B.

4. Show that if G is simple, with $n \geq k+1$, and $\delta(G) \geq (n+k-2)/2$, then G is k-connected.

   Solution: Suppose there exists an A, B separator with $k-1$ vertices. Then A or B has at most $\frac{n-k+1}{2}$ vertices and every vertex in this set has degree at most $\frac{n-k+1}{2} - 1 + k - 1 = \frac{n+k-3}{2}$, a contradiction to $\delta(G)$.

5. Show that a connected graph G is a complete graph if and only if G does not contain any induced subgraph isomorphic to $2K_2$ (i.e. just two disjoint edges) or a P_3 (a path on 3 vertices).

   Solution: $\Leftarrow$: Proof by induction on the number of vertices. Suppose G is a graph that does not contain $2K_2$ nor P_3. Then $G - x$ is complete where $x \in V(G)$. If x is not adjacent to any pair of vertices of $G - x$, then we have a $2K_2$; if x is not adjacent to any vertex of $G - x$, then we have a P_3.

6. Show that the vertex connectivity of d-dimensional hypercube is d.

   Solution: Since a d-dimensional hypercube is d-regular, deleting all neighbours of a vertex disconnects the graph, hence $\kappa \leq d$. We now prove by induction on d that between every pair of vertices in H_d, there are d internally vertex disjoint paths: let the two copies of H_{d-1} in H_d be H and H'. If vertices x, y belong to H, then there are already $d-1$ internally vertex disjoint (x, y)-paths in H. Let x', y' be the neighbours of x, y in H'. Then $(x, (x', y')$-path in $H', y)$ is a new internally vertex disjoint path. Suppose $x \in H$ and $y' \in H'$. Let $y_1, \ldots, y_d$ be the neighbours of y in H. Let $y'_1, \ldots, y'_d$ be their neighbours respectively in H'. Then d internally vertex disjoint (x, y')-paths are *(i)* $((x, y_1)$-path in $H, y, y')$ *(ii)* $((x, y_i)$-path in $H, y'_i, y')$ for $i = 2, d$. *(iii)* $(x, (x', y'_1)$-path in $H, y')$.

7. Let G be an undirected k-regular graph for an odd integer k, and let its edge connectivity be at least $k-1$. Then show that G has a perfect matching.

   Solution: Suppose G does not have a perfect matching, then let $S \subset V$ be a 'bad' set according to Tutte's theorem. Since degree of every vertex in an odd component of $G - S$ is k, the number of edges between any odd component and S must be odd. Also, due to edge connectivity, the number of edges between any component and S must be at least $k-1$. Since $k-1$ is even, there must be at least k edges between every component and S. Hence the total number of edges between all odd components and $S \geq q(G-S)k > |S|k$. But every vertex of S has degree k hence the number of edges incident on S can be at most $|S|k$.

Colouring

1. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. Consider a graph G defined as follows: $V(G)$ is a subset of $X \times Y$ and two distinct vertices $[x_i, y_j], [x_a, y_b] \in V(G)$ are adjacent if and only if: either $x_i = x_a$ or $y_j = y_b$. For $x_i \in X$, let $\gamma(x_i) = |(\{xi\} \times Y) \cap V(G)|$ and for $y_j \in Y$, let $\gamma(y_j) = |(\{yj\} \times X) \cap V(G)|$. Now get an expression for the chromatic number $\chi(G)$ of G in terms of the function γ. Prove your answer.

Solution: Construct a graph H such that $V(H) = X \cup Y$ and $x_i y_j \in E(H)$ iff $[x_i, y_j] \in V(H)$. Then two edges of H are adjacent iff two vertices of G are adjacent. Thus G is the line graph of H and $\chi(G) = \chi'(H)$. H is a bipartite graph and we know its edge chromatic number is $\Delta(H) = max\{\gamma(x_i), \gamma(y_j)\}$ over all i, j.

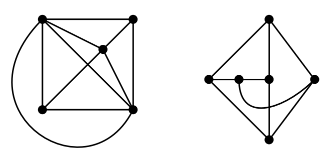
2. Let $K_{n,n,n}$ denote the complete tri-partite graph with n vertices in each part.
   (a) When $n \geq 1$ is an odd integer, what is the edge chromatic number of $K_{n,n,n}$, i.e. $\chi'(K_{n,n,n})$? Prove your answer.
   (b) Find a proper edge coloring of $K_{2,2,2}$ using 4 colors.
   (c) Show that $\chi'(K_{n,n,n}) = 2n$, when $n \geq 2$ is even. (Hint: You may try to use part (b) of this question.)

   Solution:

   (a) The total number of edges is $3n^2$. Each colour class is a matching and can hence contain at most $\frac{3n-1}{2}$ edges. From Vizing's theorem, if $\Delta = 2n$ colours are enough, then at most $\frac{3n-1}{2} \times 2n = 3n^2 - n$ edges are coloured. Hence $\chi'(G) = 2n + 1$.
   (b) $K_{2,2,2}$ can be decomposed into 4 edge-disjoint perfect matchings, each matching obtaining a colour.
   (c) Each partite set can be viewed as two sets containing $n/2$ vertices each. This overall structure resembles that of $K_{2,2,2}$. Since $\chi'(K_{2,2,2}) = 4$, each colour class of $(K_{2,2,2})$ can be replaced with a set of $n/2$ colours, hence the edges of $K_{n,n,n}$ can be coloured with $4 \times \frac{n}{2}$ colours (it might help to see that $\chi'(K_{n/2,n/2}) = n/2$).

3. Consider a drawing G' of a (not necessarily planar) graph G in the plane. Two edges of G' cross if they meet at a point other than a vertex of G'. Each such point is called a crossing of the two edges. The crossing number of G, denoted by $cr(G)$, is the least number of crossings in a drawing of G in the plane. Show that, $cr(K_5) = 1$ and $cr(K_{3,3}) = 1$.

   Solution: K_5 and $K_{3,3}$ with one crossing each:

4. Show that the crossing number of a graph G satisfies the inequality $cr(G) \geq m - 3n + 6$ provided $n \geq 3$.

   Solution: Draw a subgraph of G with maximum number of edges m' such that the subgraph is planar. Then $m' = 3n + 6$. For each of the $m - m'$ edges that we introduce back into the graph, there is at least one crossing, else the edge would already be drawn in the subgraph. Hence $cr(G) \geq m - m'$.

5. Let G be a connected planar graph with girth k, where $k \geq 3$. Show that the number of edges, $m \leq k(n-2)/(k-2)$.

   Solution: Every face has at least k edges hence $e \geq fk/2 \Rightarrow f \leq 2e/k$. The result follows from Euler's formula $n - e + f = 2$.

6. Consider the vertex coloring problem: we need to give a color to each vertex in the graph making sure that no two adjacent vertices get the same color. Given a graph G, the chromatic number of G is defined to be the minimum positive integer k such that we can color the vertices of G with k colors as described above. Show that if G is a co-comparability graph (i.e. the compliment of a comparability graph) then the biggest complete subgraph (clique) in G has exactly k vertices.

   Solution: We need to prove that $\omega(G) = \chi(G)$. This is equivalent to proving that $\alpha(\overline{G}) = \theta(\overline{G})$. Since $\overline{G}$ is a comparability graph, it does not contain a directed cycle. Hence $(V(\overline{G}), <)$ is a partially ordered set. We use Dilworth's theorem: in any finite partially ordered set, the maximum number of elements in any antichain equals the minimum number of chains in any partition of the set into chains. This corresponds to the size of a maximum independent set equals the smallest clique cover number, hence $\alpha(\overline{G}) = \theta(\overline{G})$.

7. Consider the d-dimensional hypercube H_d. Recall that the vertices of H_d corresponds to the 2^d, d-dimensional $0-1$ vectors, and two vertices are adjacent if and only if the hamming distance of the corresponding vectors is exactly 1. Show that H_d is non-planar for each $d \geq 4$.

   Solution: If $d \geq 4$, then we can find two disjoint copies of H_3 in H_d. On suitable contraction of vertices of one copy of H_3, we obtain a K_4; we contract the other copy of H_3 to a single vertex, which is adjacent to every vertex of the K_4, thus obtaining a K_5 minor. Hence H_d is non-planar.

8. What is the chromatic number of H_d, the d-dimensional hypercube?

   Solution: Let a vertex of H_d belong to A_1 if its hamming weight is odd, and A_2 if even. Any two vertices of A_i have a hamming distance of at least two and hence are not adjacent. Thus H_d is bipartite and $\chi(H_d) = 2$.

Special classes of graphs

1. Let G be an interval graph: that is to each vertex $v \in V(G)$, we can associate an interval $I(v)$ on the real line such that two vertices u and v are adjacent if and only if $I(v) \cap I(u) \neq \emptyset$. Show that $\chi(G) = \omega(G)$, where $\omega(G)$ is the clique number of G.

   Solution: Order the vertices according to the left endpoints of their intervals and colour greedily. A vertex gets the kth colour only if it is adjacent to $k-1$ coloured neighbours. Since each vertex is associated with an interval, all of its neighbours are adjacent to each other, hence forming a k-clique.

2. Let G be a non-trivial simple graph with degree sequence $(d_1, d_2, \ldots, d_n)$ where $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose that there is no integer $k < (n+1)/2$ such that $d_k < k$ and $d_{n-k+1} < n-k$. Show that G has a hamiltonian path.

   Solution: In G, for every $k < \frac{n+1}{2}$ if $d_k < k$ then $d_{n-k+1} \geq n-k$. Let G' be the graph obtained from G by adding a vertex and making it adjacent to all vertices of G. Then in G' we have if $d_k < k+1$ (i.e $d_k \leq k$) then $d_{n-k+1} \geq n-k+1$. Hence from Chvátal's condition, G' is Hamiltonian and thus G contains a Hamiltonian path.

3. A graph G is called self-complementary if it is isomorphic to $\overline{G}$, its complement. Give an example of a self-complementary graph.

Solution: P_4, C_5.

4. Prove that every self-complementary graph has a hamiltonian path.

   Solution: Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G. We observe for self-complementary graphs that a vertex with degree d_i in G has degree d_{n-i+1} in $\overline{G}$. Let G' be the graph obtained from G by adding a vertex and making it adjacent to all vertices of G. Then $(d_1+1, d_2+1, \ldots, d_n+1, n)$ is the degree sequence of G'. We prove that Chvátal's condition holds for G' : suppose $\exists v \ni d_{\overline{G}}(v) = d_i + 1 \leq i$ and $i < \frac{n+1}{2}$. Then $d_i \leq i-1$ and hence $d_{n-i+1} \geq n-i$. Hence $d_{n-i+1} + 1 \geq n-i+1$ and thus G' is Hamiltonian, which implies that G contains a Hamiltonian path.

5. Let $P = \{p_0, p_1, \ldots, p_{n-1}\}$ be a set of n distinct points on the plane. Let $r_0, \ldots, r_{n-1}$ be positive real numbers. Let (p_i, r_i) represent the circle centered at p_i and of radius r_i. Let us define a simple graph $G = (V, E)$ with $|V| = 2n$ as follows: Let $V = \{v_0, v_1, \ldots, v_{2n-1}\}$ and $f : V \to P$ be such that $f(v_i) = p_i \bmod n$. In G let v_i and v_j (for $i \neq j$) be adjacent if and only if the two circles (p_i, r_i) and (p_j, r_j) intersect. Let G' be the induced subgraph of G on the vertex set $V' = \{v_0, v_1, \ldots, v_{n-1}\} \subset V$. Show that G is a perfect graph if and only if G' is a perfect graph. (Give a complete argument).

   Solution: $\Leftarrow$: It only needs to be seen that G is an expansion of G' where vertex v_i is expanded to the edge $v_i v_{i+n}$ $(0 \leq i \leq n-1)$. Recall a lemma by Lovász which states that any graph obtained from a perfect graph by expanding a vertex is again perfect.

6. A graph G is a self-complementary graph iff G is isomorphic to its complement $\overline{G}$. Show that any regular self-complementary graph has a hamiltonian path.

   Solution: We have $\delta(G) = \frac{n-1}{2}$ as G is regular and self-complementary. Let G' be the graph obtained from G by adding a vertex and making it adjacent to all vertices of G. Then $\delta(G') = \frac{n+1}{2}$. Hence by Dirac's theorem, G' contains a Hamiltonian cycle, thus G contains a Hamiltonian path.

7. Let G be a simple graph of minimum degree δ. Show that G contains a path of length 2δ if G is connected and $\delta \leq (n-1)/2$.

   Solution: On the contrary, let $(x = v_1, v_2, \ldots, v_{2\delta} = y)$ be a longest path in G (the path contains at most $n-1$ vertices, let z be a vertex not in the path). Every neighbour of x and y must be on the path, else we can produce a longer path. Then $\exists i \ni x \leftrightarrow v_i$ and $y \leftrightarrow v_{i-1}$ and hence $(v_i, v_{i+1}, \ldots, y, v_{i-1}, v_{i-2}, \ldots, x, v_i)$ is a cycle. Since z is adjacent to some v_j, deleting the edge $v_j v_{j+1}$ we obtain a a longer path, a contradiction.

Miscellaneous

1. Show that every automorphism of a tree fixes a vertex or an edge.

   Solution: Consider a tree $T(V, E)$. Let T' be a copy of T and let f be an automorphism from T to T'. Then the leaf set of T is equal to the leaf set of T'. Deleting these leaf sets produces two subtrees say T_1 and T_1' respectively. We repeatedly delete leaf sets until we are left with either a single vertex or a single edge.

2. Show that every 2-connected graph contains a cycle.

Solution: If some 2-connected graph did not contain a cycle, then it is a tree, which contains a cut vertex (the vertex adjacent to a leaf vertex), a contradiction.

3. Show that a tree without a vertex of degree 2 has more leaves than other vertices.

   Solution: Let i be the number of internal vertices and l be the number of leaves. Then the degree sum over all vertices is at least $3i + l \leq 2(n-1)$ since there are exactly $n-1$ edges. Thus $3i + l \leq 2(i + l - 1) \Rightarrow i + 2 \leq l$ hence the result.

4. Show that a graph is bipartite iff every induced cycle has even length.

   Solution: $\Rightarrow$: Since it is not possible to partition the vertex set of an odd cycle into two independent sets, a bipartite graph cannot contain an odd cycle.
   $\Leftarrow$: Let x be a vertex of a connected graph without induced odd cycles. Let $d(v)$ be the distance between x and v for every vertex v. Let $A = \{v|d(v) \text{ is even}\}$ and $B = \{v|d(v) \text{ is odd}\}$. Then an edge in either A or B produces an odd cycle. If it not induced, then a chord of the cycle divides it into two cycles: one of even length and one of odd length. We repeatedly check if these odd cycles have a chord or not until we obtain a chordless odd cycle, a contradiction as every induced cycle is even.

5. Let $\mathcal{T}$ be a set of sub-trees of a tree. Assume that the trees in $\mathcal{T}$ have pairwise non-empty intersections. Show that their overall intersection is non-empty.

   Solution: Let the tree in question be a rooted tree. For every subtree in $\mathcal{T}$, we identify a vertex of the subtree which has the smallest depth, and designate it as the root of the subtree. Among all such roots, consider the subtree whose root has the largest depth. Every other tree must contain this root, hence the overall intersection is at least this root.