Fixed-parameter algorithms - Branching
NIT Calicut, July 2019

1. (a) A graph is a cluster graph if each connected component of the graph is a clique. Show that a graph is a cluster graph if and only if it does not have an induced path on three vertices. I.e. it does not have three vertices x, y and z such that (x, y) is an edge, (y, z) is an edge and (x, z) is NOT an edge.

   (b) Give an $O(m+n)$ algorithm to determine whether a given graph is a cluster graph. The algorithm should find an induced path on three vertices if the graph is NOT a cluster graph.

2. Give an $O^*(2^k)$ algorithm for finding a k-sized vertex cover in a graph. Can you improve the algorithm? (Hint: Branch using the observation that if a vertex is not in the solution, all its neighbors must be; so branch by picking a vertex or all its neighbours.)

3. Given a family $\mathcal{F}$ of subsets of a finite universe U, a hitting set S is a subset of U that has a non-empty intersection with every set in $\mathcal{F}$. I.e. $S \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Given a family $\mathcal{F}$ of subsets of a universe U where each set in $\mathcal{F}$ is of size at most d, and an integer k, we want to determine whether it has a hitting set of size at most k. Show that this problem is fixed-parameter tractable by giving an $O^*(d^k)$ algorithm.

4. (a) Let Π be a class of graphs. Π is called hereditary if it is closed under induced subgraphs. I.e. if $G \in \Pi$ and H is an induced subgraph of G, then H is also in Π. Is the class of all bipartite graphs hereditary? Come up with other examples of hereditary class of graphs. What is an example of a non-hereditary class of graphs?

   (b) A forbidden set F of graphs of a hereditary class Π is a set of graphs such that G is not in Π if and only some $H \in F$ is an induced subgraph of G. What is the forbidden set of graphs for the class of cluster graphs? for the class of edgeless graphs? for the class of bipartite graphs?

(c) Suppose Π is a hereditary class of graphs that has a forbidden set of size d; i.e. the forbidden set has d graphs. Consider the following problem. Given a graph G and an integer k, does G have a subset of k vertices whose deletion results in a graph in Π. Show that this problem is fixed-parameter tractable.

5. An undirected graph G is called perfect if for every induced subgrph H of G, the size of the largest clique in H is the same as the chromatic number of H. Show that a perfect graph has an odd lengthed cycle if and only if it has a triangle. Consider the ODD CYCLE TRANSVERSAL problem, restricted to perfect graphs, that is given a perfect graph G and an integer k, does there exist a set of at most k vertices whose deletion makes the graph bipartite. Design a $3^k.n^{\mathcal{O}(1)}$ algorithm for this problem (here n is the number of vertices in G).

6. In the MIN-2-SAT problem, we are given a 2-CNF formula ϕ and an integer k, and the objective is to decide whether there exists an assignment for ϕ that satisfies at most k clauses. Show that MIN-2-SAT problem is NP-complete and can be solved in $2^k n^{\mathcal{O}(1)}$ time.