Tutorial Sheet - More on Branching and Kernelization
NIT Calicut, July 2019

1. Show that any graph on n vertices with minimum degree 3 has a cycle of length at most $2\log n+1$. Can you also find such a cycle in $O(m+n)$ time?

2. Show that $(\log n)^k \leq (k\log k)^k+n$ for all positive integers n and $k \leq n$.

3. A tournament is a directed graph with a directed edge between every pair of vertices.

   (a) Show that a tournament has a directed cycle if and only if it has a triangle.

   (b) Consider the directed feedback vertex set problem where we are given a directed graph G, and an integer k and the goal is to determine whether there are k vertices whose removal results in a directed acyclic graph. Suppose the given directed graph is a tournament. Give a $O^*(3^k)$ algorithm for the problem.

4. (a) In the graph coloring problem, given a graph, and an integer k we want to determine if its vertices can be colored with at most k colors so that the end points of every edge gets different colors. What would be a brute force algorithm for the problem, and what is its running time? Is it possible to get a $f(k)n^c$ time for the problem, where c is independent of k?

   (b) In the Dual Coloring problem, we are given a graph G on n vertices and a positive integer k, and the objective is to test whether there exists a coloring of its vertices with at most $n-k$ colors such that the end points of an edge gets different colors. Obtain a kernel with $\mathcal{O}(k)$ vertices of this problem using crown decomposition (and crowm lemma). Hint: Consider a crown decomposition in the complement of G and then infer what this decomposition looks like in the original graph, to design a reduction rule that deletes the head and crown of the crown decomposition.

5. In this problem, we will try to obtain a kernel for the d-hitting set problem. A sunflower with k petals and a core Y is a collection of sets $S_1, S_2, \dots S_k$ such that $S_i \cap S_j = Y$ for all $i \neq j$; the sets $S_i \setminus Y$ are petals and we require none of them to be empty. Note that a family of pairwise disjoint sets is a sunflower (with an empty core).

   (a) Let A be a family of sets (without duplicates) over a universe U, such that each set in A has cardinality exactly d. Then show that if $|A| > d!(k-1)^d$, then A contains a sunflower with k petals and such a sunflower can be computed in time polynomial in $|A|, |U|$ and k. (Try induction on d.)

   (b) Can you come up with a reduction rule if there is a sunflower with $k+1$ petals in the given d-hitting set instance?

   (c) Obtain a kernel for the d-hitting set problem with $O(d!k^d)$ sets.