1. On a graph $G(V, E)$, define the following:

   1. A vertex cover C in G is a subset of vertices such that, for each edge $uv \in E$ either $u \in C$ or $v \in C$ (or both). The decision problem VC is: Given a) a graph $G(V, E)$ and b) a positive integer c, to determine whether G contains a vertex cover of size *less than or equal to c*. Show that VC$\in$ NP.
   2. A clique K in G is a subset of vertices such that for each $u, v \in K$, both $uv \in E$. The decision problem CLIQUE is: Given a) a graph $G(V, E)$ and b) a positive integer k, to determine whether G contains a clique of size *greater than or equal to k*. Show that CLIQUE$\in$ NP.
   3. An independent set I in G is a subset of vertices such that for each $u, v \in I$, $uv \notin I$. The decision problem IS is: Given a) a graph $G(V, E)$ and b) a positive integer i, to determine whether G contains an independent set of size *greater than or equal to i*. Show that IS$\in$ NP.
   4. Design polynomial time reduction algorithms between each pair of these problems.

2. The Hitting Set Problem (HS) is defined as follows. Given a) a non-empty set U, b) a positive integer m, c) a collection of non-empty subsets $S_1, S_2, \ldots S_m$ of U and d) a positive integer h. The question is to determine whether there exists a subset H of U such that H contains at most h elements and $H \cap S_i \neq \emptyset$ for all $1 \leq i \leq m$. That is, H must "hit" every set S_i. Show that the problem is NP-complete.

3. The Set Cover Problem (SC) takes as instance a) non-empty set U, b) a positive integer m, c) a collection of non-empty subsets $S_1, S_2, \ldots S_m$ of U and d) a positive integer k. The question is to determine whether we can choose a k or fewer sets from among $S_1, S_2, \ldots S_m$ such that the union of the k selected sets is U. That is, the selected sets must "cover" U. Show that the problem is NP-complete.

4. Integer Linear Constraint Satisfiability Problem (ILCS): A Constraint system is specified by a set of linear inequalities. For instance, the set $\{2x + 5y \geq 3, 3x - 4y \leq 4\}$ forms a constraint system of two inequaliies in two variables. The question is whether a given constraint system is integraly satisfiable - that is, whether it has an **integer solution** (for example, $x = 1, y = 1$ is an integer solution to the above constraint system). Show that the problem of determining whether a given constraint system is satisfiable is NP complete.

5. Let L, L' be languages. In each case determine whether the language L i) is in P, ii) is in NP, iii) is NP-hard iv)is NP-complete. (There are three possibliies in each case - "Yes", "No" or "No conclusion can be drawn from the given assumpions").

   1. $L' \in$ P and $L \preceq_p^m L'$.
   2. $L' \in$ NP and $L \preceq_p^m L'$.
   3. $L' \in$ NP and $L' \preceq_p^m L$.
   4. $L' \notin$ P and $L' \preceq_p^m L$.
   5. $L' \in$ NP complete, $L' \preceq_p^m L$ and $L \preceq_p^m L'$.

6. Let L, L' be languages. In each case determine whether i) P = NP ii) P $\neq$ NP or iii) neither of the above conclusions are possible from the given assumptions.

   1. $L \in$ P, $L \preceq_p^m L'$, $L' \in$ NP. Will the conclusion change if L' is NP complete?
   2. $L \in$ NP, $L \preceq_p^m L'$, $L' \in$ P.
   3. L is NP hard, $L \preceq_p^m L'$, $L' \in$ P. Will the conclusion change if L is NP-complete?
   4. $L \notin$ P, $L \preceq_p^m L'$, $L' \in$ NP.