- Vertex Cover
- Cluster Vertex deletion
- Feedback Vertex Set
- d-hitting set
- Odd Cycle transversal

Branching

Kernelization

Iterative Compression

Hardness theory in Parameterized Complexity

Known hard problem Π_1

$$\Big\downarrow \text{p-time}$$

current problem Π_2

Consequence: If my current-problem Π_2 is in P, so is Π_1

$\Pi_1 \quad (x, k)$

$$\Big\downarrow f(k)|x|^c$$

$\Pi_2 \quad (x', k')$ function of k alone

Consequence: If Π_2 is FPT then so in Π_1.

Ind Set Problem (G, K) — G has an IS of size k

iff

Vertex Cover (G', k') — G' has a VC of size k'

$(G', k') = (G, n-k)$

Ind Set (G, k) is W-hard in the parameterized world

$\Downarrow$ (no $f(k)n^c$ algm likely)

CLIQUE is W-hard

K-coloring is unlikely to have $f(k)\, n^c$ algm

$\hookrightarrow n^{g(k)}$ algm

(para NP-hard)

Multicolor Ind Set :

Input : $(G, V_1 \cup V_2 \cup \ldots V_k)$

par : k

Qn : Does $\exists$ an IS set in G that has exactly one vertex from each V_i . ?

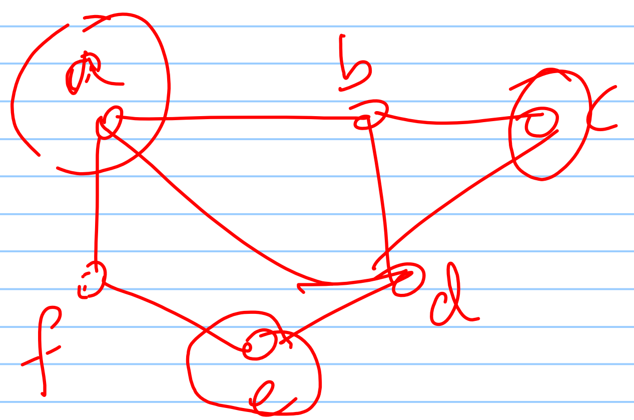
is W-hard

Dominating Set

Input : $G(V, E)$

par : k

Qn : $\exists$ a dom set of size $\leq k$?



$S \subseteq V$ is a dominating set if

$\forall v \in V$

$v \in S$

or $\exists u$ adj to v

s.t. $u \in S$

Reduction from MCIS $\longrightarrow$ Dom set

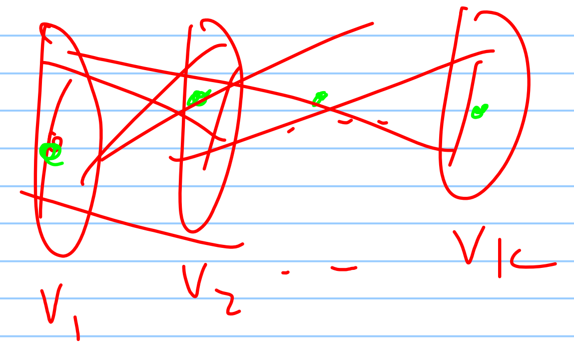
$G = (V_1 \cup V_2 \cup \dots V_k, E)$ $\longrightarrow$ G', k

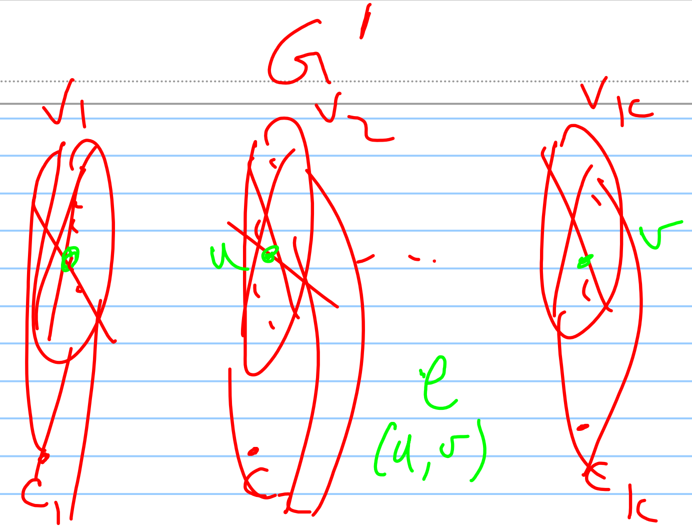
k

G has a multicolored IS of size k iff G' has a dom set of size k

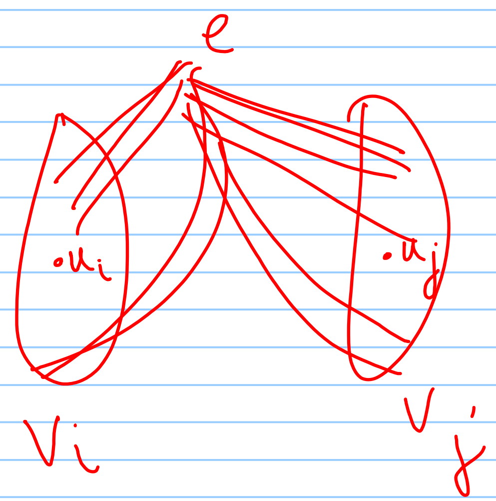
MCIS

G





$e = (u_i, u_j)$
$\in \quad \in$
$V_i \quad V_j$



Claim: G has a MCIS of size k
iff G' has a dom set of size k.

Given a CNF formula F, K

Qn: $\exists$? a satisfying assignment with at most K ones?

$$F = (x_1 \vee x_3 \vee \bar{x}_4) \wedge (x_2 \vee \bar{x}_7 \vee x_6) \wedge \cdots \wedge (\ )$$

W-hard when the clause sizes are unbounded.

Dom set $\sim$ G, k $\qquad V - v_1 \ldots v_n$

Formula F, k $\qquad$ Vars $- x_1 \ldots x_n$

$$\forall v_i \quad x_i = 1 \text{ or } \bigvee_{j \in N(i)} x_j$$

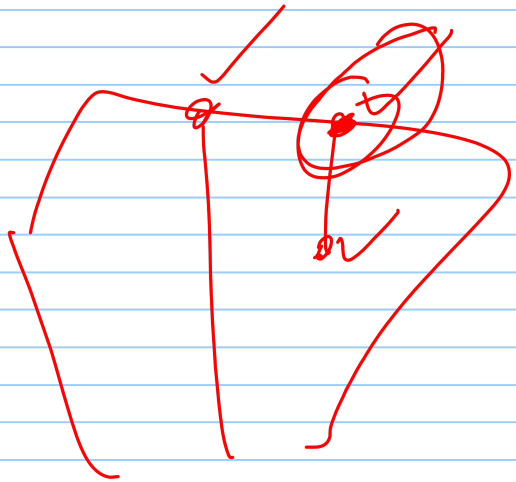
$$F = \bigwedge_{i=1}^{n} \left(x_i \vee \bigvee_{j \in N(i)} x_j \right)$$

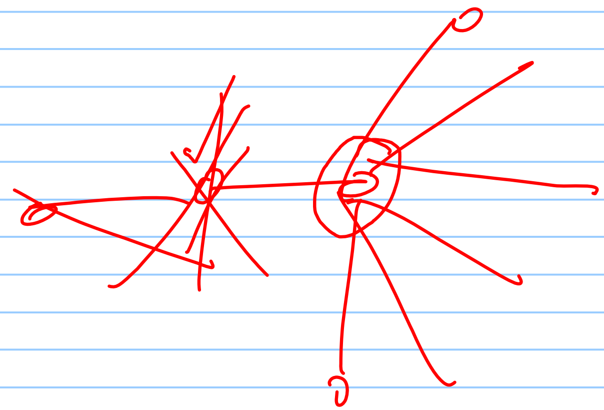
G has a dom set of size $\leq k$

iff

F has a sat asst with $\leq k$ ones.

Dominating set with degree bounded by d has an $O^*(d^k)$ algm (by branching or reducing to d-CNF wt k problem).





If G has $> k(d+1)$ vertices,

return NO

else G has $\leq k(d+1)$ vertices.

so a kernel

Exact Exponential Algorithms

(no parameters; or treat input size as the parameter).

3-SAT : given 3-CNF formula F, is it satisfiable? Trivial $2^n |F|$ algorithm

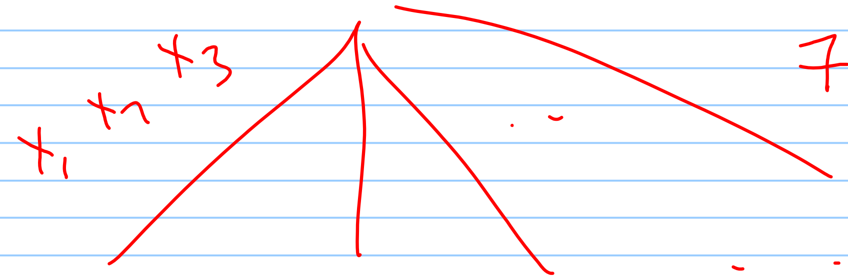
Chromatic Number: k^n if $k = \chi(G)$

We'll improve these algorithms in this lecture.

3-SAT $F = (x_1 \vee x_2 \vee \overline{x}_3) \wedge (\quad) \wedge (\quad) \ldots \wedge (\quad)$

How many assts for $x_1 \ldots x_n$ in which $x_1 = 0, x_2 = 0, x_3 = 1$

$\frac{7}{8} \times 2^n$ assts are suff to try.



7 assts

Recurse on the remaining vars

$$\begin{cases} T(1) = \Theta(1) \\ T(n) = 7T(n-3) = 7(7T(n-6)) \\ \qquad = 7^2 T(n-6) \end{cases}$$

(Recursively check if $F|_{x_1 = -,\ x_2 = -\ x_3}$ is satisfiable

or $F|_{\ -\ \ -}$

$F|_{-\ \ --}$

$$7^{n/3} = \left(7^{1/3}\right)^n \sim (1.\ldots)^n$$

3-SAT current best $\sim (1.5..)^n$

For general SAT — no $(2-\varepsilon)^n$ algm is known

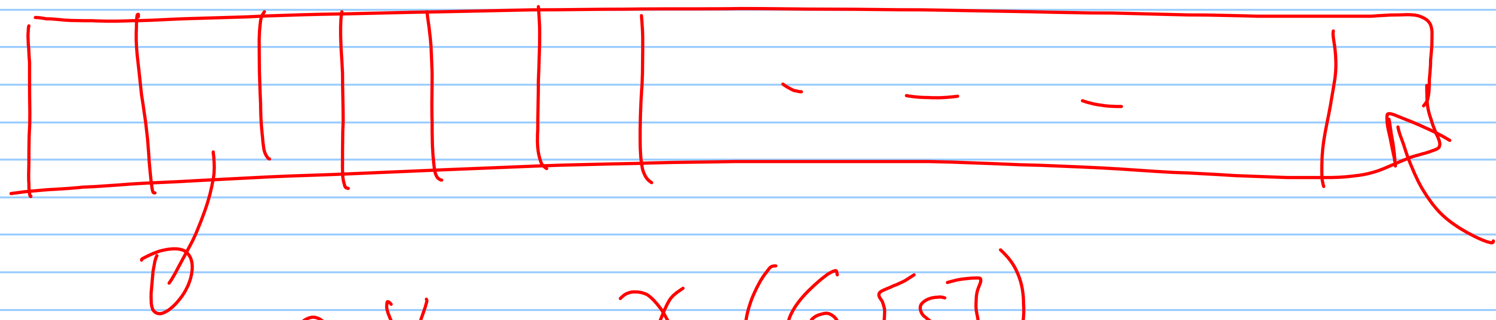
SETH $\sim$ no $(2-\varepsilon)^n$ algm possible for general SAT.

Input G

$$\chi(G) = \min_{\substack{S \subseteq V \\ S \text{ is independent}}} \{1 + \chi(G - S)\}$$

$$\underbrace{\chi(G[S])}_{\text{induced subgraph}} = \min_{\substack{S' \subseteq S \\ S' \text{ is ind in } G[S]}} 1 + \chi(G[S] - S')$$

Starting from one vertex graphs.



$S \subseteq V$, $\chi(G[S])$

2^n entries, each take $\leq 2^n$ time

so overall $O^*(4^n)$ time.

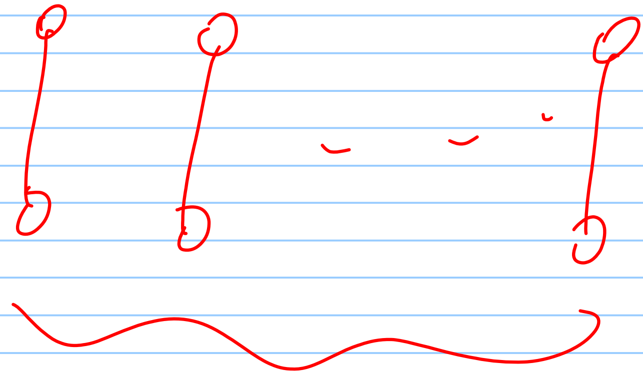
Overall time $= O^*\left(\sum_{S \subseteq V} 2^{|S|}\right)$

$$= \sum_{i=1}^{n} \binom{n}{i} 2^i \leq 3^n$$

Can be improved by

- enumerating over all maximal ind sets in each step and
- using the fact that # maximal

ind sets in a graph on n vertices is at most $3^{n/3}$.



$n/2$

Exact Exponential Algorithms

– Fomin & Kratsch