Problem Set I

1. Suppose a vector $v \in \mathbf{R^3}$ has coordinates $(1, 1, 1)$ w.r.t the basis $(1, 1, 0), (1, 0, 1), (0, 1, 1)$, Find its coordinates with respect to the basis $(1, 0, 0), (1, 1, 0), (1, 1, 1)$. Find the matrix of basis translation between these two bases. ☐

2. Consider the vector space of all polynomials of degree at most n over $\mathbf{R}$. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$. Find the co-ordinates of $f(x)$ over the basis $[1, (x-t), (x-t)^2, \cdots, (x-t)^{n-1}]$. (Hint: Evaluate $\frac{d^k}{dx^k} f(x)$ at $x = t$.) ☐

3. Let $p_1, p_2, \cdots$ be a listing of all primes ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$ etc.) Consider the vector space $\mathbf{R}$ over $\mathbf{Q}$. (In this vector space, the vectors are real numbers and scalars are rational numbers). Show that $\{p_1, p_2, \cdots\}$ is a linearly dependent set, but the set $\{\log p_1, \log p_2, \log p_2, \cdots\}$ is linearly independent. (Use the fact that every natural number has a unique factorization into prime factors.) ☐

4. Consider the linear map from $\mathbf{R^3}$ to $\mathbf{R^2}$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$. Find the matrix of this linear transformation if the basis on either side is the standard basis. What is the matrix of the map if the basis in $\mathbf{R^2}$ is changed to $(1, 1), (1, -1)$? What is the matrix when the basis for $\mathbf{R^3}$ is changed to $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ and the basis of $\mathbf{R^2}$ is the standard basis? What is the matrix of the map if the basis for $\mathbf{R^3}$ is $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ and basis for $\mathbf{R^2}$ is $(1, 1), (1, -1)$? ☐

5. In the above question, find the equations to $img(T)$ and $ker(T)$ (assume standard basis on both sides). Find a basis for $ker(T)$ and $Img(T)$. Find a collection of vectors in $\mathbf{R^3}$ whose images under T form a basis for $img(T)$. What is $Rank(T)$, $Nullity(T)$? ☐

6. Let T be a linear transformation from V to W over a field F. Let $Dim(V) = n, U = Nullspace(T)$. Let $Nullity(T) = k$ and $rank(T) = r$. Let $u_1, u_2, .., u_k$ be a basis of U. Suppose we extend this set with vectors $b_1, b_2, .., b_r$ to form a basis of V, is it always true that $T(b_1), T(b_2), .., T(b_r)$ forms a basis of $Img(T)$? Give a proof/counterexample. ☐