Problem Set III

1. Let R be a ring and I be an ideal in R. Show that $\frac{R}{I}$ is a ring with operations defined by $(a+I)+(b+I)=(a+b)+I$ and $(a+I)(b+I)=ab+I$ □

2. Let f be an onto homomorphism from a ring R to a ring R'. Let $I = ker(f)$. □

    1. Show that there is a bijective homomorphism (isomorphism) from $\frac{R}{I}$ to R'.
    2. Show that f is injective if and only if $I = \{0\}$.
    3. Show that if R is a field then either $f = 0$ or f is an isomorphism.

3. An ideal I in a ring R is principal if there exists an element $d \in I$ such that $I = dR$. That is, elements in I are obtained by multiplying each element of R with some particular element $d \in I$. Such a d satisfying $I = dR$ is called a **generator** of the ideal. □

    1. Show that every ideal in $\mathbf{Z}$ is principal.
    2. Let $a_1, a_2, \ldots a_n \in R$. Show that $I = \{a_1r_1 + a_2r_2 + \cdots a_nr_n : r_1, r_2, \ldots r_n \in R\}$ is an ideal. This ideal is called the ideal generated by $a_1, a_2, \ldots a_n$ and is denoted by $I(a_1, a_2, \ldots a_n)$.
    3. Consider the set $R[x]$ of polynomials with real coefficients. Show that every ideal I is $R[x]$ is principal. (Use the fact that Euclid's algorithm can be applied to polynomials as well).

4. Let V be a vector space of dimension n over a field F. Let W be a subspace of V of dimension k. Let $w_1, w_2, \ldots w_k$ be a basis of W. Let $v_{k+1}, v_{k+2}, \ldots v_n$ extended $w_1, w_2, \ldots w_k$ to a basis of V. Consider the quotient group $\frac{V}{W}$ (Vectors form an Abelian group w.r.t addition and W is a subgroup of V). Define scalar multiplication of $\frac{V}{W}$ as $\alpha(v+W) = \alpha v + W$. □

    1. With the definition of scalar multiplication above, show that $\frac{V}{W}$ is a vector space.
    2. Show that $v_{k+1}+W, v_{k+2}+W, \ldots v_n+W$ is a basis of $\frac{V}{W}$. Hence conclude that $dim(\frac{V}{W}) = dim(V) - dim(W)$.

5. Let $T : V \mapsto V'$ be an surjective linear tranformation. Let $W = ker(T)$. Show that there is a bijective map $\overline{T} : \frac{V}{W} \mapsto V'$ (define $\overline{T}(v+W) = T(v)$). Show that $\overline{T}$ is indeed a linear tranformation. Using this observation (called the homomorphism Theorem for Vector Spaces) and the previous question, deduce the rank-nullity theorem. □

6. Let m and n be positive integers with $n > m$. Consider $f : \mathbf{Z}_n \mapsto \mathbf{Z}_m$ defined by $f(x) = x \mod m$. Show that f is a homomorphism if and only if m divides n. When f is a homomorphism, what is $ker(f)$? □

7. Suppose m, n be positive integers. □

    1. Show that $\mathbf{Z}_m \times \mathbf{Z}_n$ is cyclic with generator $(1,1)$ if and only if GCD$(m,n) = 1$. When
    2. If GCD$(m,n) = 1$, show that the map $f : \mathbf{Z}_{mn} \mapsto \mathbf{Z}_m \times \mathbf{Z}_n$ defined by $f(x) = (x \mod m, x \mod n)$ is an isomorphism.
    3. If GCD$(m,n) \neq 1$, and f is defined as in the previous sub-question, what is $ker(f)$?