Problem Set V

1. If A is a complex $n \times n$ matrix such that $det(A) = 0$, then, show that the function $f : \mathbf{C}^n \times \mathbf{C}^n \mapsto \mathbf{C}$ defined by $f(u, v) = u^T A \overline{v}$ does not define an inner product. Which inner product axiom is violated in this case? ☐

2. In R^2 consider the positive definite matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Consider the inner product in $\mathbf{R}^2$ defined by A. ☐

   1. Run Gram Schmidt orthogonalization (starting from the standard basis) to find a basis that is orthonormal for the inner product.
   2. Find the LU decomposition of A.
   3. Find the coordinates of the vector $[1, 1]^T$ with respect to the basis you have found.

3. [Bessel's inequality] Let $b_1, b_2, \ldots b_k$ be orthogonal unit vectors in an n dimensional complex inner product space V with inner product function $()$. Let $v \in V$. Show that $||v||^2 \geq \sum_{i=1}^{k} P_i(v)^2$ where $P_i(v) = (v, b_i)b_i$ is the projection of v along the direction b_i. ☐

4. If U is an $n \times n$ unitary matrix, show that the columns of A form an orthonormal basis of $\mathbf{C}^n$. ☐

5. Let $d = [d_1, d_2, \ldots d_n]^T$ (d is a column vector) be a **unit** vector in $\mathbf{R}^n$. Show that the $n \times n$ matrix $D = dd^T$ satisfy the property that for every vector $y = [y_1, y_2, \ldots y_n]^T$ in $\mathbf{R}^n$, Ay gives the orthogonal projection of y along the direction d with respect to the standard inner product in $\mathbf{R}^n$. ☐

6. Consider $\mathbf{R}^3$ with the standard inner product. Use the previous question to find the matrix of the orthogonal projection operator (with respect to the standard basis) that project each vector to ☐

   1. The direction along the direction determined by $[1, 1, 1]^T$.
   2. The subspace that is orthogonal to the direction defined by $[1, 1, 1]^T$.
   3. The plane $x + y + z = 0$.

7. Let $b_1, b_2, \ldots b_n$ be an orthonormal basis of a complex vector space $\mathbf{C}$. Find the matrix (w.r.t the basis $b_1, b_2, \ldots b_n$) of the orthogonal projection operator that projects each vector on to the subspace spanned by b_1 and b_2. ☐

8. Find the distance from the vector $[5, 5, 5]$ to the plane defined by $x + 2y + 3z = 0$? ☐

9. Consider $\mathbf{R}^2$ with the standard inner product. Let T be an operator in $\mathbf{R}^2$ whose matrix w.r.t the standard basis is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find an orthonormal basis of $\mathbf{R}^2$ with respect to which the matrix of T becomes a diagonal matrix? ☐

10. Let V be a complex inner product space of dimension n. Let $P_1, P_2, \ldots P_k$ be operators on V satisfying a) $P_i^2 = P$ for all i and j. b) For each $u, v \in V$ and $i, j, (i \neq j)$, $(P_i(u), P_j(v)) = 0$. Let $P = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$ for **real** scalars $\lambda_1, \lambda_2, \ldots \lambda_k$. Show that for any vectors $u, w \in V$, P satisfies $(u, Pv) = (Pu, v)$. ☐