Problem Set II

1. Let V is a vector space over real numbers. Let $() : V \times V \to \mathbf{R}$ be a function satisfying the inner product axioms - that is, for all $u, v, w \in V$, $a, b \in \mathbf{R}$, a) $(u, av + bw) = a(u, v) + b(u, w)$, b) $(u, v) = (v, u)$, c) $(u, u) \geq 0$ and $(u, u) = 0$ if and only if $u = 0$. Define $||u|| = \sqrt{(u, u)}$ and $d(u, v) = ||u - v||$. For the following questions, assume that u, v and w are abitrary vectors. ☐

    1. Show that if $u, v \in V$ such that $(u, v) = 0$ then $||u + v||^2 = ||u||^2 + ||v||^2$.
    2. Show that $|(u, v)| \leq ||u||||v||$ (Cauchy Schwartz inequality).
    3. Show that $d(u, v) + d(v, w) \geq d(u, w)$.

2. Let $()$ be the standard dot product on $\mathbf{R}^3$. Find two mutually perpendicular unit vectors u_1 and u_2 that are also perpendicular to the vector $v = [1, 1, 1]^T$. (Don't start drawing pictures! translate the requirement to an algebraic condition involving $()$ and solve). ☐

3. Let $()$ be the standard dot product on $\mathbf{R}^3$. Find a unit vector u perpendicular to both $v_1 = [1, 1, 0]^T$ and $v_2 = [0, 1, 1]^T$. ☐

4. Let $()$ be the standard dot product on $\mathbf{R}^2$. Let $b_1 = [1, 1]^T$ and $b_2 = [1, -1]^T$. Suppose $v = 2b_1 + 3b_2$. Find the coordinates of a unit vector u with respect to the basis $[b_1, b_2]$such that $(u, v) = 0$. ☐

5. Let $()$ be the standard dot product on $\mathbf{R}^2$. Consider the basis, $b_1 = [1, 1]^T$ and $b_2 = [1, 0]^T$ of $\mathbf{R}^2$. Find the 2×2 matrix A such that if $v = x_1 b_1 + x_2 b_2$ and $w = y_1 b_1 + y_2 b_2$, then $(u, w) = [x_1, x_2]A[y_1, y_2]^T$. Solve the same question with $b_1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$ and $b_2 = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]^T$ ☐

6. Consider the vector $v = (1, 1, 1, 1)$ in $\mathbf{R}^4$. Let $u = (1, 1, 0, 0)$. Find a vector w perpendicular to u such that $v - w$ is a vector along the direction of u. ☐

7. Let A be any $n \times n$ real matrix. Show that $x^T A^T A x \geq 0$ for any vector $x \in \mathbf{R}^n$. Further show that if A is non singular, then A is a symmetric positive definite matrix. ☐

8. Let B be any non singular $n \times n$ real matrix. Let $A = B^T B$. For any vectors $x, y \in \mathbf{R}^n$, define $(x, y) = x^T A y$. Show that the function $()$ so defined satisfies all the inner product axioms. ☐

9. Define the inner product $()$ function on $\mathbf{R}^2$ as follows: Given vectors $u = [x_1, x_2]^T$ and $v = [y_1, y_2]^T$ in $\mathbf{R}^2$, define $(u, v) = [x_1, x_2] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Find a basis b_1, b_2 of $\mathbf{R}^2$ such that for any vectors $u = x_1 b_1 + x_2 b_2$ and $v = y_1 b_1 + y_2 b_2$, then $(u, v) = x_1 y_1 + x_2 y_2$. (Basically, the question asks you to find a basis b_1, b_2 of $\mathbf{R}^2$ with respect to which $()$ behaves just like the standard dot product). ☐

10. Find a non-singular 2×2 real matrix B such that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = B^T B$. (Can you see the connection to the previous question!?) ☐