Problem Set III

1. Let V be an inner product space of dimension n with inner product function $()$. Let S be a subspace of V of dimension k for some $0 < k < n$. Define $S^{\perp} = \{u \in V : (u, s) = 0 \text{ for all } s \in S\}$. $S^{\perp}$ is called the **orthogonal complement** of the subspace S (with respect to the inner product $()$). (The more common name for $S^{\perp}$, "perpendicular space of S" is used mostly when the vector space V is $\mathbf{R}^n$ and the inner product under consideration is the standard inner product.) ☐

    1. Show that $S^{\perp}$ is a subspace of V.
    2. Show that if $v \in V$ satisfies $v \in S \cap S^{\perp}$ then $v = 0$. (Thus 0 vector is the only vector in the intersection of S and $S^{\perp}$.)
    3. Suppose $s, s' \in S$ and $w, w' \in S^{\perp}$. Suppose $s + w = s' + w'$ then show that $s = s'$ and $w = w'$. This shows that if any vector $v \in V$ can be written as the sum of two vectors - one in the subspace S and one perpendicular to S, then there is only one way to do so.
    4. Suppose $v = s + w$ for some $s \in S$ and $w \in S^{\perp}$. Show that $||v||^2 = ||s||^2 + ||w||^2$. (Pythagoras theorem).

2. Let V be an inner product space of dimension n with inner product function $()$. Let S be a subspace of V of dimension k for some $0 < k < n$ as in the previous question. Let $b_1, b_2, \ldots b_k$ be an orthonormal basis for S. Let $v \in V$. Let $\alpha_i = (v, b_i)$. ☐

    1. Show that $w = v - \sum_{i=1}^{k} \alpha_i b_i \in S^{\perp}$.
    2. Show that we can write every vector $v \in V$ as $v = s + w$ for some unique $s \in S$ and $w \in S^{\perp}$. (The vector $s \in S$ is called the **orthogonal projection** of v on to S and w is called the component of v orthogonal (perpendicular to) to S. These two questions essentially work out the theory of how to resolve a vector $v \in V$ into components along and perpendicular to S.
    3. Show that $dim(S^{\perp}) = n - k$. (Hint: Assume $dim(S^{\perp}) = m$ and consider any orthonormal basis $c_1, c_2, \ldots c_m$ of $S^{\perp}$. Prove that $b_1, b_2, \ldots b_k, c_1, c_2, \ldots c_m$ is a basis of V.)

3. This, and the next questions puts the theory of orthogonal projections developed in the last two questions into practical problem solving. Consider the subspace S of $\mathbf{R}^3$ spanned by the vectors $[1, 1, 1]^T$ and $[1, 1, 0]^T$. Assume the standard inner product. Resolve the vectors $v_1 = [0, 0, 1]^T$ and $v_2 = [1, 1, 2]^T$ into components along S and perpendicular to S. Find the coordinates (with respect to the standard basis in terms of x, y and z) of the components of the generic vector $v = [x, y, z]^T$ in $\mathbf{R}^3$ along and orthogonal to S. ☐

4. Define the inner product $()$ function on $\mathbf{R}^2$ as follows: Given vectors $u = [x_1, x_2]^T$ and $v = [y_1, y_2]^T$ in $\mathbf{R}^2$, define $(u, v) = [x_1, x_2] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Consider the vector $e_1 = [1, 0]^T$. Let S be the subspace of $\mathbf{R}^2$ spanned by e_1 (essentially the x-axis). Find the components of the vector $[2, 1]^T$ along S and orthogonal to S. (Hint: In component resolution problems of the kind presented in this question as well as the previous question, first find an orthonormal basis for S (w.r.t. the inner product under consideration) and the first two questions show that this makes component resolution easy.) ☐

5. Let V be an inner product space of dimension n with inner product function $()$. Let S be a subspace of V of dimension k for some $0 < k < n$. Let $v \in V$ be any vector. Let s and w be the components of v along and orthogonal to S. Let $s' \in S$. Show that $d(v, s') \geq d(v, s)$. (Hint: $d^2(v, s') = ||v - s + s - s'||^2$ - why?) This result shows that the component of v along s is the nearest (point of shortest distance) from v in the subspace S, and is called the **approximation theorem**. ☐

    Approximation theorem explains why orthogonal projections are used for dimensionality reduction (projecting a data vector of real numbers with lots of components into a lower dimensional space of fewer components to reduce the sample size of the data set so that computation becomes manageable) in the field of big data analysis. Orthogonal projection gives the nearest (with respect to Euclidean distance) approximation of the given data vector in the lower dimensional subspace. The following questions develop some matrix based calculation techniques that makes problem solving easy. First we introduce some notation.

Let V be a real inner product space with inner product function $()$. A subspace S of dimension 1 in V is sometimes called a *direction*. To specify a direction in V, sometimes a unit vector $d \in S$ satisfying $span(d) = S$ is specified (with the convention that the direction intended is $span(d)$). In this case d is called a **unit direction**. Let v be an arbitrary vector in V. The projection of v along the subspace $S = Span(d)$ is called the **projection of v along the direction** d.

6. In R^3 (with standard inner product), consider the unit direction vector $d = [\frac{\sqrt{3}}{2}, \frac{1}{2}, 0]$. The 3×3 matrix P is defined by $P_d = dd^T$. Let $v = [x, y, z]^T$. Show that the vector $P_d[x, y, z]^T$ is the orthogonal projection v along the subspace . This shows that computation of the projection along the direction d can be transformed into matrix multiplication. The following question shows that projection into higher dimensional subspaces can also be transformed into matrix multiplication. ☐

7. In $\mathbf{R}^3$ (with standard inner product), consider the mutually perpendicular direction vectors $c = [\frac{\sqrt{3}}{2}, \frac{1}{2}, 0]$ and $d = [-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0]$. Let $P_c = cc^T$ and $P_d = dd^T$. Let $S = span(c, d)$ be the two dimensional subspace of $\mathbf{R}^3$ spanned by c and d. ☐

   1. Find the orthogonal projection of the vector $[3, 1, 1]$ along the subspace S.
   2. Show that the orthogonal projection of the vector $[x, y, z] \in \mathbf{R}^3$ into S is given by $(P_d + P_c)[x, y, z]^T$.

   The next question generalizes the matrix based computational technique seen in these problems. (The question looks complicated, but it is just a general statement for the computation done in the last two questions).

8. Let V be an inner product space of dimension n with inner product function $()$. Let $b_1, b_2, \ldots b_n$ be an orthonormal basis of V. Let d be a unit direction vector such that $d = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$. Let $S = span(d)$. Define the matrix $P_d = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T[\alpha_1, \alpha_2, \ldots, \alpha_n]$. Let v be any vector in V such that $v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n$. Let s be the orthogonal projection of v onto the subspace S. Let $[y_1, y_2, \ldots y_n]^T$ be the coordinate vector s with respect to the basis $[b_1, b_2, \ldots b_n]$. (i.e., $s = y_1 b_1 + y_2 b_2 + \cdots + y_n b_n$). Show that $[y_1, y_2, \ldots y_n]^T = P_d[x_1, x_2, \ldots, x_n]^T$. ($P_d$ is called the **projection matrix** for the direction d with respect to the basis $[b_1, b_2, \ldots, b_n]$.) ☐

   Suppose $d' = \alpha'_1 b_1 + \alpha'_2 b_2 + \cdots + \alpha'_n b_n$ an another unit vector **orthogonal to** d, Let $P'_d = [\alpha'_1, \alpha'_2, \ldots, \alpha'_n]^T[\alpha'_1, \alpha'_2, \ldots, \alpha'_n]$. Let $S' = span(d, d')$. Show that $(P_d + P'_d)[x_1, x_2, \ldots, x_n]^T$ gives the projection of v onto the subspace S'.

9. Using Gram Schmidt orthogonalization, find a 2×2 upper triangular matrix U such that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = U^T U$. ☐