MFCS September 2018; Name:

1. Give an example of a 3×3 real matrix A that is not the identity matrix such that every non-zero vector in $\mathbf{R}^3$ is an Eigen vector of A. Justify your answer. [3]

   Soln: For $A = kI$, every non-zero $x \in \mathbf{R}^3$ is an Eigen vector with Eigen value k.

2. Let G be a complete graph of 5 vertices. Let A be the adjacency matrix of A. What will be the sum of the Eigen values of A? Justify your answer. [3]

   Soln: The sum of the Eigen values of $A =$ trace $(A) = 0$.

3. Let $T : \mathbf{R}^3 \mapsto \mathbf{R}^3$ satisfy $T(i) = i$, $T(i+j) = 2(i+j)$ and $T(i+j+k) = 3(i+j+k)$. Find a basis of Eigen vectors of T. What will be the matrix of T with respect to this basis? [3]

   Soln: $b_1 = i, b_2 = (i+j), b_3 = (i+j+k)$ are Eigen vectors with Eigen values $1, 2$ and 3 respectively. The matrix of T w.r.t this basis will be the 3×3 diagonal matrix with $1, 2$ and 3 on the diagonal.

4. Let S be a subspace of an inner product space V. Let $s \in S$. Let $t \in S^{\perp}$. Show that s and t are linearly independent. [3]

   Soln: Let scalars α, β be such that $\alpha s + \beta t = 0$. Then we have, $0 = (s, \alpha s + \beta t) = \alpha(s, s)$ Since $s \neq 0$, $(s, s) \neq 0$. Hence $\alpha = 0$. Similarly, $\beta = 0$.

5. Let V be a real vector space and T an operator on V. Let a, b, c be three distinct Eigen values of T. Let u, v, w be the corresponding Eigen vectors. Show that u, v, w are linearly independent. [3]

   Soln: Let $\alpha u + \beta v + \gamma w = 0$ for some scalars α, β, γ. Without loss of generality we may assume that $\gamma \neq 0$. Multiplying the equation with c, we get $c\alpha u + c\beta v + c\gamma w = 0$. Then $T(\alpha u + \beta v + \gamma w) = a\alpha u + b\beta v + c\gamma w = 0$. Subtracting this equation from the previous one, we get $(c-a)\alpha u + (c-b)\beta v = 0$. Since both $c - a$ and $c - b$ are non-zero, it follows that u and v, which are Eigen vectors corresponding to distinct Eigen values a and b are linearly dependent, which is a contradiction. (We had proved in the class that Eigen vectors corresponding to distinct Eigen values must be linearly independent).

6. Let A be an $m \times n$ real matrix. Argue that $Rank(A) \leq \min\{m, n\}$. (You may assume theorems/results proved in the class, but state which theorems/results are used while using them). [3]

   Soln: As A is a linear map from $\mathbf{R}^n \mapsto \mathbf{R}^m$, by Rank Nullity Theorem $CRank(A) \leq m$. Since A^T is a linear map from $\mathbf{R}^m$ to $\mathbf{R}^n$, $CRank(A^T) = RRank(A) \leq n$. The result follows from the fact that $RRank(A) = CRank(A)$.