Mid Semester I **CS 4050 Algorithms Part II** **Marks:15**

Name and Roll No.: ____________________

1. How many permutations of the set $\{1, 2, .., n\}$ have the property that the i^{th} element is the smallest among the first i elements? Justify your answer (no credits without justification). $\boxed{3}$

   Solution: For each among the $C(n, i)$ choices of the i elements, $(i - 1)!$ permutations which keep the smallest among those in the i^{th} position. The remaining elements may be permuted in $(n - i)!$ ways. Thus total $C(n, i)(i - 1)!(n - i)! = \frac{1}{i}n!$ permuations.

2. Suppose X is a random variable with $E(X) = \mu$. Is it always true that $E(X^2) \geq \mu^2$? Justify your answer. $\boxed{3}$

   Solution: $f(x) = x^2$ is convex everywhere on the real line. Hence $E(X^2) \geq E^2(X)$ by Jensen's inequality.

3. Consider the proposal algorithm for the stable marriage problem running with n men and n women. How many proposals suffice before it is guarenteed that at least one women is permanently married? Derive an asymptotically tight upper bound on the number of proposals necessary for the algorithm to terminate. $\boxed{3}$

   Solution: A female shall receive up to n proposals from different men before getting married permanently. Since there are only n women, the total number of proposals is bounded by n^2.

4. Suppose there are n balls numbered $\{1, 2, ..n\}$ and n bins also numbered $\{1, 2, ..., n\}$. Suppose each ball is thrown into a random bin. What is the expected number of balls that would land in a bin with the same number? (No credits without proper analysis and justification). $\boxed{3}$

   Solution: For $1 \leq i \leq n$, let X_i denote the random variable that takes value 1 if the i^{th} ball lands in the i^{th} bin and 0 otherwise. Since $Pr(X_i = 1) = \frac{1}{n}$, $E(X_i) = 1/n$. Let $X = \sum_i X_i$. The required expectation is $E(X) = \sum_i E(X_i) = \sum_i \frac{1}{n} = 1$.

5. Let G be the complete graph with n vertices. Suppose we start a *random walk* from a vertex v in G. At each subsequent step, we choose a random neighbour of the current vertex and move to that vertex. What is the expected number of steps required for the random walk to visit every vertex in the graph? $\boxed{3}$

   Solution Let X_i denote the time (number of steps) to have $i + 1$ vertices visited after the visit to the i^{th} vertex. $X_1 = 1$ and for $2 \leq i \leq n - 1$, X_i follows geometric distribution with parameter $\frac{(n-1)-(i-1)}{n-1} = \frac{n-i}{n-1}$. This is because of the following: suppose you have visited i vertices already, the next hop from one of these vertices is to an unvisited neighbour with probability $\frac{(n-1)-(i-1)}{n-1}$. Thus $E(X_i) = \frac{n-1}{n-i}$ by geometric distribution. The expected number of steps to visit every vertex at least once is $E(\sum_i X_i) = \sum_i E(X_i) = \sum_i \frac{n-1}{n-i} = (n - 1)\sum_i \frac{1}{i} = \Theta(n \log n)$.