Assignment II

Design and Analysis of Algorithms

1. Let L_1, L_2 be two languages such that $L_1 \leq_p^m L_2$. In each of the following cases, ascertain whether you can conclude that $P = NP$ or $P \neq NP$.
   1. $L_1 \notin P$, $L_2 \in NP$.
   2. $L_1 \in NP$, $L_2 \notin NP$.
   3. $L_1 \in P$, $L_2 \notin NP$.
   4. $L_1 \notin P$, $L_2 \notin NP$.
   5. $L_1 \in P$, $L_2 \in NP$.
   6. $L_1 \in NP$, $L_2 \in P$.
   7. L_1 is NP complete, $L_2 \in NP$, but not NP complete.
   8. $L_1 \in P$, L_2 is NP complete.
   9. $L_1 \notin NP$, L_2 is NP complete.
2. Define by $coNP = \{L : \Sigma^* \setminus L \in NP\}$. $L \in coNP$ is said to be $coNP$ complete if for all $L' \in coNP$, $L' \leq_p^m L$. Show that L is NP complete if and only if $\Sigma^* \setminus L$ is $coNP$ complete. Show that $P \subseteq NP \cap coNP$. It is not known whether $NP = coNP$.
3. Show that if any language in $NP \cap coNP$ is NP complete, the $NP = coNP$.
4. Show that $2CNFSAT$ is polynomial time solvable. (Hint: Construct a graph G with one node each for every variable x_i and its negation $\neg x_i$ in the given formula. For each clause of the form $(x_i \vee x_j)$, add an edge from $\neg x_i$ to x_j and $\neg x_j$ to x_i in the graph...etc. Now, the problem of addressing consistancy (satisfiability) of the given clauses reduces to a cycle checking problem in the graph constructed).
5. Show that $2CNFSAT \leq_p^m CLIQUE$. In view of the fact proved in the previous question, can you conclude anything about the P vs NP conjucture from this reduction?
6. The Subgraph Isomorphism problem SI takes as input two graphs G_1 and G_2 and asks whether G_1 contains a subgraph Isomorphic to G_2. Show that $CLIQUE \leq_p^m SI$. (Give the reduction algorithm - specify the input and output clearly). Show that SI is NP complete.
7. The set cover problem is specified by ground set U, a collection of subsets $S_1, S_2, .., S_m$ of U and a positive integer $k \leq m$. The question is to determine whether it is possible to pick k sets from $S_1, S_2, .., S_m$ whose union is U. Show that the set cover problem is NP complete. (Hint: Reduction from vertex cover).
8. Suppose in the set cover problem above, it is given that each element in U appears in at most t sets, show that by through an LP formulation of the set cover problem and then by rounding an optimal fractional LP solution, we can get a factor t approximation algorithm for the problem.
9. A dominating set S in a graph $G(V, E)$ is a subset of vertices such that for every vertex v in V, either $v \in S$ or there must be some neighbour of $u \in S$ such that $(u, v) \in E(G)$. The dominating set problem DS takes as input a graph $G(V, E)$ and a positive integer k and asks whether G contains a dominating set of k vertices. Show that the problem is NP complete. (Note the similarity with vertex cover).
10. The traveling salesman's problem (TSP) takes as input a weighted undirected complete graph $G(V, E)$ with $w(u, v)$ denoting the weight of the edge connecting u and v. The problem is to determine a tour of minimum cost in G such that it starts from any vertex in G, visits every vertex once, and comes back to the starting vertex. This problem is not only NP complete, but also hard to approximate (it is known that if we can find a "good" approximation algorithm to this problem, then $P = NP$). However, if the edge weights satisfy the triangular inequality (as in the case of the Metric tree problem: $w(u, v) \leq w(u, x) + w(x, v)$ for all $x, u, v \in V$), show that a 2 approximation algorithm is possible. (Hint: The approach is similar to the metric Steiner tree problem. Note that the cost of a minimum spanning tree in G is a lower bound to the cost of the optimal salesman tour (why?)).