Lovasz's Perfect Graph Theorem

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. We view an elements of $\mathbb{R}_+^V$ as vectors, but use function notation, so if $x \in \mathbb{R}_+^V$ and $v \in V$ we write $x(v)$ instead of x_v. For $S \subseteq V$ we define $x(S) = \sum_{v \in S} x(v)$.

Independent set polytopes: For every graph $G = (V, E)$ we define the polytopes

$$P(G) = \{x \in \mathbb{R}_+^V : \ x(K) \leq 1 \text{ for every clique } K\}$$
$$P_I(G) = \{x \in \mathbb{R}_+^V : x \text{ is a convex combination of characteristic vectors of independent sets}\}.$$

Integral: A polytope is *integral* if every vertex has all coordinates integers.

Observation 1 *$P(G)$ is integral if and only if $P(G) = P_I(G)$.*

Proof: This follows from the observation that the integral points in $P(G)$ are precisely the characteristic vectors of independent sets. (Thus $P_I(G) \subseteq P(G)$, and $P(G) = P_I(G)$ if and only if every vertex of $P(G)$ is a vertex of $P_I(G)$.) $\square$

Observation 2 *If $P(G)$ is integral and $X \subseteq V(G)$, then $P(G \setminus X)$ is integral.*

Proof: It suffices to show that $P(G \setminus v)$ is integral for an arbitrary vertex $v \in V(G)$. To see this, note that $P(G \setminus v)$ is precisely the intersection of $P(G)$ with the hyperplane $\{x \in \mathbb{R}^V : x(v) = 0\}$. Since $x(v) \geq 0$ is a constraint of $P(G)$, it follows that $P(G \setminus v)$ is a face of $P(G)$. It is an immediate consequence of this that $P(G \setminus v)$ is integral. $\square$

Perfect Graphs: We say that a graph G is perfect if $\omega(G \setminus X) = \chi(G \setminus X)$ for every $X \subseteq V(G)$. Note that if G is perfect, then $G \setminus Y$ is perfect for every $Y \subseteq V(G)$.

Replication: Let G be a graph and let $v \in V(G)$. To *replicate* v, we add a new vertex v' to the graph add an edge between v' and every neighbor of v and then add an edge between v' and v.

Lovasz Weightings: If $w \in \mathbb{Z}_+^V$, we let G_w be the graph obtained from G by deleting every vertex v with $w(v) = 0$ and replicating each vertex v with $w(v) > 0$ exactly $w(v) - 1$ times (note that the resulting graph does not depend on the order of operations).

Lemma 3 *If G is perfect and $w \in \mathbb{Z}_+^V$, then G_w is perfect.*

Proof: Since G_w is obtained from a sequence of vertex deletions and replications, it suffices to show that if the graph G' is obtained from G by replicating the vertex v, then G' is perfect. To prove this, it is enough to show that $\chi(G') = \omega(G')$ since for any induced subgraph of G' a similar argument works. In fact, $\chi(G') \geq \omega(G')$ trivially, so we need only prove that $\chi(G') \leq \omega(G')$. If v is contained in a maximum clique of G, then we have $\omega(G') = \omega(G) + 1 = \chi(G) + 1 \geq \chi(G')$. Thus, we may now assume that v is not contained in a maximum clique of G. Next, let $\omega = \omega(G)$, choose a colouring of G with colour classes $A_1, A_2, \ldots, A_\omega$, and assume that $v \in A_\omega$. Since the graph $G \setminus (A_\omega \setminus \{v\})$ is a perfect graph which has no clique of size ω (why!), we may choose a colouring of this graph with colour classes $B_1, B_2, \ldots, B_{\omega-1}$. Now, $B_1, B_2, \ldots, B_{\omega-1}, A_\omega$ is a list of independent sets in G which use v twice, and every other vertex once. By replacing one occurence of v with v', we get a colouring of G' with ω colours. Thus $\chi(G') \leq \omega = \omega(G')$ and we are finished. □

Theorem 4 (Lovasz's Perfect Graph Theorem) *For every graph $G = (V, E)$, the following are equivalent.*

(i) *G is perfect.*

(ii) *$P(G)$ is integral.*

(iii) *$\bar{G}$ is perfect.*

Proof: It suffices to show (i) ⇒ (ii) ⇒ (iii), since $\bar{\bar{G}} = G$ then yields (iii) ⇒ (i).

(i) ⇒ (ii): To prove (ii), we shall show that $P(G) = P_I(G)$. Let $x \in P(G) \cap \mathbb{Q}^V$. Now it suffices to show $x \in P_I$. Choose a positive integer N so that $w = Nx \in \mathbb{Z}^V$, and consider the graph G_w. For every $i \in V$, let Y_i be the set of vertices in G_w which are equal to i or obtained by replicating i and let $\pi : V(G_w) \to V$ be given by the rule that $\pi(u) = i$ if $u \in Y_i$. Let $\tilde{K}$ be a maximum size clique in G_w and let $K = \pi(\tilde{K})$. Then, K is a clique of G and further,

$$\omega(G_w) = |\tilde{K}| \leq \sum_{i \in K} |Y_i| = w(K) = Nx(K) \leq N$$

(here the last inequality follows from $x \in P(G)$). Since G_w is perfect, we may choose a colouring of it with colour classes $A_1, A_2, \ldots, A_N$. Now, consider $\pi(A_1), \pi(A_2), \ldots, \pi(A_N)$.

This is a list of independent sets in G which use every vertex $i \in V$ exactly $w(i)$ times. It follows from this that $x = \frac{1}{N} w = \frac{1}{N} \sum_{\ell=1}^{N} \mathbb{1}_{\pi(A_\ell)}$ where $\mathbb{1}_A$ is the characteristic vector of $A \subseteq V$. Thus $x \in P_I$ as desired.

(ii) $\Rightarrow$ (iii): It follows from Observation 2 that property (ii) holds for any subgraph obtained from G by deleting vertices. In light of this, it suffices to prove $\chi(\bar{G}) = \omega(\bar{G})$. We shall prove this by induction on $|V|$. As a base, observe that the result holds for the trivial graph. Let $\alpha = \alpha(G)$ be the size of the largest independent set in G. Since $P(G)$ is integral (i.e. $P(G) = P_I(G)$), every vertex of $P(G)$ is the characteristic vector of an independent set. It follows from this that $F = P(G) \cap \{x \in \mathbb{R}^V : x^\top 1 = \alpha\}$ is a face of $P(G)$. Consider a generic point x in the face F. There must be a constraint of the form $x(K) \leq 1$ which is tight for x (otherwise, the only tight constraints are nonnegativity constraints, and we could freely increase any positive coordinate of x while staying in $P(G)$ - which is contradictory). Now the constraint $x(K) \leq 1$ must be tight for every point in F, and it follows that the clique K has nonempty intersection with every independent set of G of size α. This gives us $\alpha(G \setminus K) = \alpha(G) - 1$ or equivalently, $\omega(\bar{G} \setminus K) = \omega(\bar{G}) - 1$. By induction, we may choose a colouring of $\bar{G} \setminus K$ using $\omega(\bar{G}) - 1$ colours. Adding the set K (which is independent in $\bar{G}$) to this, gives us a colouring of $\bar{G}$ using $\omega(\bar{G})$ colours, thus completing the proof. $\square$