Assignment I

Question 1: Let f be a function from set X to set Y. Let A, B be subsets of X and P, Q be subsets of Y. Let $\overline{A} = X \setminus A$, $\overline{Q} = Y \setminus Q$ etc.

1. Show that $f(A \cup B) = f(A) \cup f(B)$.
2. Given an example showing that $f(A \cap B) \neq f(A) \cap f(B)$
3. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$. Show that equality holds if f is injective.
4. Show that $f(A) \setminus f(B) \subseteq f(A \setminus B)$.
5. Give an example showing that $f(A) \setminus f(B) \neq f(A \setminus B)$. Show that equality holds if f is injective.
6. $f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$.
7. $f^{-1}(P \cap Q) = f^{-1}(P) \cap f^{-1}(Q)$.
8. $f^{-1}(P \setminus Q) = f^{-1}(P) \setminus f^{-1}(Q)$.
9. $A \subseteq f^{-1}(f(A))$.

Question 2: Let R be a relation on a set X. Define $R^0 = I = \{(x, x) : x \in X$. Inductively define $R^{i+1} = \{(x, y) : \exists z \in X$ such that $(x, z) \in R$ and $(z, y) \in R^i\}$. Define the reflexive transitive closure of R, $R^* = I \cup R \cup R^2 \cup ....$

1. Show that R^* is transitive.
2. Suppose R' is another **transitive** relation on X such that $R \subseteq R'$. (That is, R' must be transitive, moreover for all $(x, y) \in R$, it must be true that $(x, y) \in R'$ as well.) Show that $R^* \subseteq R'$. This result shows that any transitive relation that is an extension of R must contain R^*. In other words, R^* is the smallest transitive relation that extends R.
3. Show that if X has n elements, $R^* = I \cup R \cup R^2 \cup ... \cup R^{n-1}$. That is, one needs to find the union of only the first $n-1$ terms in the definition of transitive closure.

Question 3: Let R be an equivalence relation on a set X. Let $x \in X$. Define $R(x) = \{y | (x, y) \in R\}$.

1. Suppose $x, x' \in X$ such that $R(x) \cap R(y) \neq \emptyset$, show that $R(x) = R(y)$.

2. Show that $\bigcup_{x \in R} R(x) = X$

3. Show that if $X_1, X_2, ..X_m$ are subsets of X such that $X_i \neq \emptyset$ and $X_i \cap X_j = \emptyset$ whenever $i \neq j$, then the relation $R = \{(x, x') | \exists X_i$ such that $x \in X_i$ and $x' \in X_i\}$ is symmetric and transitive. What additional condition on $X_1, X_2, ..X_m$ is required for R to become reflexive?

4. For any positive integer n, show that the relation $R = \{(a, b) | (a - b)$is a multiple of $n\}$ defined on the set of integers (denoted by $\mathbf{Z}$) is an equivalence relation. Find the partitions defined by this equivalence relation.

Question 4: Let X be any set. Consider the set 2^X (power set of X). Let g be any *injective* function from S to 2^X. That is, for each element $x \in X$, $g(x)$ (we will write g_x instead $g(x)$) is subset of X such that for each distinct $x, x' \in X$, g_x and $g_{x'}$ are distinct subsets of X. The objective of this question is to show that g can't be surjective.

1. Consider the following subset of X, $S = \{x \in X | x \notin g_x\}$. That is an element x is present in S if and only if x is not a member of the set g_x.

2. Show that $S \neq g_x$ for any $x \in X$. Hence $S \notin$ Image(g). Thus g is not surjective.

3. Show that there do exist an injective map from X to 2^X. Hence argue that for any set X, $|X| < |2^X|$.

Question 5 Let X and Y be sets

1. if f is an injective map from X to Y, show that we can find a surjective map from Y to X.

2. if f is a surjective map from X to Y, construct an injective map from Y to X.

3. Let $\mathbf{Z}$ be the set of integers. Find a map f from $\mathbf{Z}$ to $\mathbf{Z}$ which is injective but not surjective and another map g from $\mathbf{Z}$ to $\mathbf{Z}$ which is surjective but not injective. (The existance of an injective map and another surjective map between two sets does not immediately imply that we can find a bijective map between the sets. This fact known as **Schoder Bernstein Theorem** will be proved in class).