Exercise I

1. In $\mathbf{R}^2$ a vector has cordinates $(2, 1)$ with respect to the basis $[1, 1]^T$, $[0, 1]^T$. What are the cordinates of this vector with respect to the basis $[1, -1]^T, [1, 0]^T$? Suppose a vector has cordinates (x, y) in the former basis, find the matrix with which you have to multiply the old cordinates to get the new cordinates (α, β) with respect to the new basis. This matrix is called the *matrix of change of basis* from the first basis to the second.

2. Consider the basis $b_1 = [\cos\theta, \sin\theta]^T$, and $b_2 = [1, 0]^T$ of $\mathbf{R}^2$, where $0 < \theta < \pi/2$. Suppose you apply Gram Schmidt process to (b_1, b_2), what are the orthonormal basis vectors you will get? Suppose you apply Gram Schmidt process to (b_2, b_1), what are the orthonormal basis vectors you will get? (Think and don't start calculating..)

3. Let $b_1 = [\cos\theta, \sin\theta]^T$, $0 < \theta < \pi/2$ and b_2 be the vector obtained by roating b_1 right by 90 degree in $\mathbf{R}^2$. Let B be the matrix obtained by writing b_1 and b_2 as column vectors. What is the inverse of this matrix? (Again don't start calculating..).

4. Recall the C_{NOT} and H gates discussed in class. Consider $|\Phi >= C_{NOT}(H|0\rangle \otimes |0\rangle)$. Express $|\Phi\rangle$ in the standard basis. Suppose you measure the first bit of $|\Phi\rangle$ in the standard basis, what is the probability with which the first bit will be observed to be $|0\rangle$? What will be the superposition state of the second bit in this case? Work out the case if the first bit is observed to be $|1\rangle$ after measurement. Do you note the curious situation that *measurement of the first bit actually determintes the second bit as well*? Analyze the situation if you measure the first bit (only) of $|\Phi\rangle$ with respect to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ as was discussed in class.

5. Let T be a linear operator (transformation) in $\mathbf{R}^2$ that act as the following on the standard basis vectors:
$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
Suppose T is applied to a vector with coordinates (x, y), what will be the cordinates of the transformed vector? Can you give a matrix representation for T? (i.e., find a matrix which when multiplied by the column vector $[x, y]^T$ yields the resultant vector.) Do you observe that knowledge of T on the basis vectors suffices to find the matrix?