Tensor products in finite dimensional complex inner product spaces

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Chapter 1

Introduction

This monograph primarily focuses on introducing tensor product spaces over finite dimensional complex inner product spaces when constrained orthonormal bases. We believe that placement of these constraints over general theory helps in developing a more accessible material for computer science audience without compromising on mathematical rigour and practical applicability. Tensor product spaces are typically introduced using concepts that use abstract algebra. In this notes an intuitive approach is used by defining the set of all multi-linear functions as the tensor product space. This approach is historically well-known but owing to the generality in developing the theory, mathematicians use abstract algebra. Most of the engineering applications, in particular machine learning and quantum computation in computer science require working on finite dimensional real or complex inner product spaces. Moreover working on orthonormal basis is sufficient to work on most of the applications. Hence, we initially focus on developing the theory of tensor product spaces of complex inner product spaces limiting to orthonormal bases. For those mathematically inclined or interested more in general theory can refer appendix to understand the theory of tensor product spaces over any finite dimensional vector spaces. The exposition is written in an incremental manner in order to gain more intuition into the k-fold tensors theory.
We expect reader to be familiar with basic notions in linear algebra. In this section, some properties of orthonormal bases of finite dimensional complex inner product spaces are shown which help in simplifying proofs provided in the next chapter. Notice that this theory works for vector spaces over real fields also.

1.1 Prerequisites

Definition 1.1.1. Let V be a finite dimensional vector space over field $\mathbb{C}$. A function $() : V \times V \to \mathbb{C}$ is called an inner product if it satisfies the following,
Linearity : $\forall\ x, y, z \in V$, $\forall\ \alpha \in \mathbb{C}$,

$$(x, y + z) = (x, y) + (x, z)$$

$$(x, \alpha y) = \alpha\,(x, y)$$

Conjugate symmetry : $\forall\ x, y \in V$,

$$(x, y) = \overline{(y, x)}$$

Positive definiteness : $\forall\ x \in V$,

$$\text{if } x \neq 0 \text{ then } \ (x, x) > 0$$

Remark :

1. $\forall\ x, y, z \in V$, $(x + y, z) = (x, z) + (y, z)$

$$(x + y, z) = \overline{(z, x + y)} = \overline{(z, x)} + \overline{(z, y)} = (x, z) + (y, z)$$

2. $\forall\ x, y \in V$, $\forall\ \alpha \in \mathbb{C}$, $(\alpha x, y) = \overline{\alpha}\,(x, y)$

$$(\alpha x, y) = \overline{(y, \alpha x)} = \overline{\alpha} \cdot \overline{(y, x)} = \overline{\alpha} \cdot (x, y)$$

3. $\forall\ x \in V$, $(0, x) = (x, 0) = 0$

$$(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0) \implies (x, 0) = 0 \implies (0, x) = \overline{(0, x)} = 0$$

4. Let $x \in V$, using the above remark 3, we get $(x, x) = 0 \iff x = 0$

Let $\dim(V) = n < \infty$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Then, $\forall\ i, j \in \{1, 2, ..., n\}$,

$$\begin{aligned}(a_i, a_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j\end{aligned}$$

Lemma 1.1.1. Let V be a finite dimensional inner product space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. $\forall$ $x \in V$,

$$\boxed{x = \sum_{i=1}^{n} (a_i, x)\, a_i}$$

Proof. $\forall\ x \in V$, since A is a basis of V, there exist unique $\alpha_i \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n} \alpha_i a_i \tag{1.1}$$

$\forall\ j \in \{1, 2, ..., n\}$,

$$(a_j, x) = \left(a_j, \sum_{i=1}^{n} \alpha_i a_i\right)$$

Using linearity of inner product we get,

$$(a_j, x) = \sum_{i=1}^{n} (a_j, \alpha a_i) = \sum_{i=1}^{n} \alpha_i (a_j, a_i)$$

Since A is an orthonormal basis of V we get,

$$(a_j, x) = \alpha_j \tag{1.2}$$

Combining equations 1.1 and 1.2 we get,

$$x = \sum_{i=1}^{n} (a_i, x)\, a_i$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. $\forall\ x \in V$ using Lemma 1.1.1 we get,

$$\boxed{x = \sum_{i=1}^{n} (a_i, x)\, a_i = \sum_{j=1}^{n} (b_j, x)\, b_j}$$

We set the convention that the coordinates of any vector $x \in V$ with respect to a basis A is a **column vector**. Formally,

Definition 1.1.2. Let V be a finite dimensional vector space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be a basis of V. $\forall\ x \in V$ there exist unique $\alpha_i \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n} \alpha_i a_i = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{bmatrix}$$

We denote the coordinates of the vector x with respect to basis A as follows,

$$^{A}x = \begin{bmatrix} \alpha_1 & \alpha_2 & . & . & \alpha_n \end{bmatrix}^T \in \mathbb{C}^n$$

We denote the r-th coordinate of vector x with respect to basis A as follows,

$$^{A}x\,[r] = \alpha_r \in \mathbb{C}$$

Remark :

1. If A is an orthonormal basis of V. $\forall\ x \in V$ using Lemma 1.1.1 we get,

$$x = \sum_{r=1}^{n} (a_i, x)\, a_i \implies {}^{A}x\,[i] = \alpha_i = (a_i, x)\ \ \forall\ i \in \{1, 2, ..., n\}$$

Lemma 1.1.2. Let V be a finite dimensional inner product space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. $\forall$ $x, y \in V$,

$$\boxed{(x, y) = \sum_{r=1}^{n} \overline{{}^{A}x\,[r]} \cdot {}^{A}y\,[r]}$$

Proof. $\forall$ $x, y \in V$ since A is an orthonormal basis of V there exist unique α_r, $\beta_s \in \mathbb{C}$ such that,

$$x = \sum_{r=1}^{n} \alpha_r a_r \qquad\qquad y = \sum_{s=1}^{n} \beta_s a_s$$

$$\begin{aligned}
(x, y) &= \left(\sum_{r=1}^{n} \alpha_r a_r, \sum_{s=1}^{n} \beta_s a_s \right) \\
&= \sum_{r=1}^{n} \sum_{s=1}^{n} \overline{\alpha_r} \cdot \beta_s \cdot (a_r, a_s) \\
&= \sum_{r=1}^{n} \overline{\alpha_r} \cdot \beta_r \qquad \text{(since } A \text{ is an orthonormal basis of } V\text{)}
\end{aligned}$$

We already know that,

$${}^{A}x\,[r] = \alpha_r \qquad\qquad {}^{A}y\,[r] = \beta_r$$

$$\implies (x, y) = \sum_{r=1}^{n} \overline{{}^{A}x\,[r]} \cdot {}^{A}y\,[r]$$

□

Definition 1.1.3. Let V be a finite dimensional inner product space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. $\forall$ $x, y \in V$ we define the dot product of x and y with respect to basis A as

$$\boxed{{}^{A}x \odot {}^{A}y = \sum_{r=1}^{n} \overline{{}^{A}x\,[r]} \cdot {}^{A}y\,[r] = \left({}^{A}x\right)^{*} \cdot \left({}^{A}y\right)}$$

Remark :

1. Lemma 1.1.2 implies that the dot product of any two vectors $x, y \in V$ has the same value irrespective of the choice of orthonormal basis. More concretely, Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. $\forall\ x, y \in V$ using Lemma 1.1.1 and Lemma 1.1.2 we get,

$$^{A}x \odot {}^{A}y = (x, y) = {}^{B}x \odot {}^{B}y$$

Definition 1.1.4. A matrix $M \in \mathbb{C}^{n\times n}$ is called non-singular if $\forall\ \alpha \in \mathbb{C}^n$,

$$M \cdot \alpha = 0 \implies \alpha = 0$$

Remark :

1. Several equivalent definitions for the non-singularity of a matrix M can be found in various textbooks. In the above definition we define M to be non-singular if and only if $Nullspace\,(M) = \{0\}$[1].

Theorem 1.1.3. Let V be a finite dimensional vector space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two bases of V. Then there exists a non-singular matrix $M \in \mathbb{C}^{n\times n}$ such that,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

Note that the matrix M is called the **basis transformation matrix** from A **to** B.

Proof. Since B is a basis of V $\forall\ j \in \{1, ..., n\}$ there exist unique $M_{ij} \in \mathbb{C}$ such that,

$$a_j = \sum_{i=1}^{n} M_{ij} b_i = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} \begin{bmatrix} M_{1j} \\ M_{2j} \\ . \\ . \\ M_{nj} \end{bmatrix}$$

[1]To explore more about Null space and Singularity refer [3] or [7]

$$\implies \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & . & . & M_{1n} \\ M_{21} & M_{22} & . & . & M_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ M_{n1} & M_{n2} & . & . & M_{nn} \end{bmatrix}$$

Let $M = \begin{bmatrix} M_{11} & M_{12} & . & . & M_{1n} \\ M_{21} & M_{22} & . & . & M_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ M_{n1} & M_{n2} & . & . & M_{nn} \end{bmatrix} \in \mathbb{C}^{n \times n}$. Then,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

It remains to show that M is non-singular. $\forall\ \alpha \in \mathbb{C}^n$,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} \cdot \alpha = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M \cdot \alpha$$

If $M \cdot \alpha = 0$ then

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} \cdot \alpha = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M \cdot \alpha = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} 0 = 0$$

Let $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & . & . & \alpha_n \end{bmatrix}^T$. Then,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{bmatrix} = 0 \implies \sum_{i=1}^{n} \alpha_i a_i = 0$$

Since A is a linearly independent set we get,

$$\alpha_i = 0\ \forall\ \{1, 2, ..., n\} \implies \alpha = 0$$

$\square$

Corollary 1.1.4. Let V be a finite dimensional vector space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two bases of V. Let $M \in \mathbb{C}^{n\times n}$ be the basis transformation matrix from A to B i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

$\forall\, x \in V$,

$$\boxed{{}^{B}x = M \cdot {}^{A}x}$$

Proof. $\forall\, x \in V$,

$$x = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} \cdot {}^{A}x = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} \cdot {}^{B}x$$

Since M is the basis transformation from A to B we get

$$\begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M \cdot {}^{A}x = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} {}^{B}x$$

$$\implies \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} \left({}^{B}x - M \cdot {}^{A}x\right) = 0$$

Since B is a linearly independnent set, we get

$${}^{B}x = M \cdot {}^{A}x$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two bases of V. If M is the basis transformation matrix from basis A to basis B then M^{-1} is the basis transformation matrix from B to A. Note that M^{-1} is well defined since M is non-singular. More concretely, $\forall\, x \in V$,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M \implies {}^{B}x = M \cdot {}^{A}x$$

$$\implies \boxed{{}^{A}x = M^{-1} \cdot {}^{B}x} \implies \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} M^{-1}$$

Theorem 1.1.5. Let V be a finite dimensional inner product space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. Let $M \in \mathbb{C}^{n\times n}$ be the basis transformation matrix from A to B i.e.,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

Then, M is an orthogonal matrix. That is,

$$\boxed{M \cdot M^* = M^* \cdot M = I \implies M^{-1} = M^*}$$

Proof. Let $E = \{e_1 = \begin{bmatrix} 1 & 0 & . & . & 0 \end{bmatrix}^T, e_2 = \begin{bmatrix} 0 & 1 & . & . & 0 \end{bmatrix}^T, ..., e_n = \begin{bmatrix} 0 & 0 & . & . & 1 \end{bmatrix}^T\}$. Note that E is the standard orthonormal basis of $\mathbb{C}^n$. It is easy to notice that $\forall$ $i \in \{1, 2, ..., n\}$,

$$^{A}a_i = e_i$$

Since M is the basis transformation matrix from A to B we get that $\forall$ $j \in \{1, 2, ..., n\}$,

$$^{B}a_j = M \cdot {}^{A}a_j = M \cdot e_j$$

Using Lemma 1.1.2 we get that $\forall$ $i, j \in \{1, 2, ..., n\}$,

$$(a_i, a_j) = \overline{\left({}^{B}a_i\right)} \odot \left({}^{B}a_j\right)$$

$$\implies (a_i, a_j) = \overline{(M \cdot e_i)} \odot (M \cdot e_j) = e_i^* \cdot M^* \cdot M \cdot e_j$$

Note that $\forall$ $N \in \mathbb{C}^{n\times n}$,

$$e_i^* N e_j = N_{ij}$$

$$\implies (a_i, a_j) = [M^* \cdot M]_{ij}$$

Since A is an orthonormal basis of V we get that

$$\begin{aligned} [M^* \cdot M]_{ij} &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}$$

$$\implies M^* \cdot M = I$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. If M is the basis transformation matrix from A to B then, $M^{-1} = M^*$ is the basis transformation matrix from B to A.

Definition 1.1.5. Let V be a finite dimensional inner product space over $\mathbb{C}$. $\forall$ $u \in V$ with $\|u\| = \sqrt{(u, u)} = 1$ (i.e, u is a unit vector) the projection operator is defined along u, $P_u : V \to V$ as follows, $\forall\ x \in V$,

$$\boxed{P_u(x) = (u, x)\, u}$$

Remark :

1. Note that $P_u(x) \in Span\{u\}$

Lemma 1.1.6. Let V be a finite dimensional inner product space over $\mathbb{C}$. $\forall$ $u \in V$ with $\|u\| = 1$, $\forall\ x \in V$,

$$(u, x - P_u(x)) = 0$$

(which means that $x - P_u(x)$ is orthogonal to u)

Proof. $\forall\ u \in V$ with $\|u\| = 1$, $\forall\ x \in V$, from Definition 1.1.5 we get that,

$$\begin{aligned} (u, x - P_u(x)) &= (u, x - (u, x)\, u) \\ &= (u, x) - (u, x)(u, u) \end{aligned}$$

Since $\|u\| = \sqrt{(u, u)} = 1$,

$$(u, x - P_u(x)) = (u, x) - (u, x) = 0$$

□

Lemma 1.1.7. If $A = \{a_1, a_2, ..., a_n\}$ is an orthonormal set of vectors in V, then A is a linearly independent set.

Proof. If $A = \{a_1, a_2, ..., a_n\}$ is an orthonormal set of vectors in V then, $\forall\ i, j \in \{1, 2, ..., n\}$,

$$\begin{aligned} (a_i, a_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}$$

Let $\alpha_i \in \mathbb{C}\ \forall\ i \in \{1, 2, ..., n\}$. Consider,

$$\sum_{i=1}^{n} \alpha_i a_i = 0$$

$\forall\ j \in \{1, 2, ..., n\}$,

$$0 = (a_j, a_i) = \left(a_j, \sum_{i=1}^{n} \alpha_i a_i \right) = \sum_{i=1}^{n} \alpha_i (a_j, a_i) = \alpha_j$$

$$\implies \alpha_j = 0\ \forall\ j \in \{1, 2, ..., n\}$$

$$\implies A \text{ is a linearly independent set}$$

□

Theorem 1.1.8. (Gram Schmidt Orthogonalization) Every finite dimensional inner product space over $\mathbb{C}$ has an orthonormal basis.

Proof. Let V be a finite dimensional inner product space over $\mathbb{C}$. Let $(V) = n < \infty$. Since $\dim(V) = n$ there exist a set $A = \{a_1, a_2, ..., a_n\}$ such that A spans V and A is a linearly independent set. Now we use Gram-Schmidt process to define the following vectors,

$$\tilde{b}_1 = a_1 \qquad\qquad b_1 = \frac{\tilde{b}_1}{\|\tilde{b}_1\|}$$

$a_1 \neq 0$ since A is a linearly independent set $\implies \|\tilde{b}_1\| \neq 0 \implies b_1$ is well defined.

$$\tilde{b}_2 = a_2 - P_{b_1}(a_2) \qquad b_2 = \frac{\tilde{b}_2}{\|\tilde{b}_2\|} \qquad \ldots$$

$$\tilde{b}_i = a_i - \sum_{j=1}^{i-1} P_{b_j}(a_i) \qquad b_i = \frac{\tilde{b}_i}{\|\tilde{b}_i\|} \qquad \ldots$$

$$\tilde{b}_n = a_n - \sum_{j=1}^{n-1} P_{b_j}(a_n) \qquad b_n = \frac{\tilde{b}_n}{\|\tilde{b}_n\|}$$

From Remark 1.1 it follows that $Span\{b_1, b_2, ..., b_i\} = Span\{a_1, a_2, ..., a_i\}\ \forall\ i \in \{1, 2, ..., n\}$.

For $b_2, b_3, ..., b_n$ to be well-defined it is necessary to prove the following claim,

Claim : $\forall\ i \in \{2, ..., n\}$,

$$\tilde{b}_i = a_i - \sum_{j=1}^{i-1} P_{b_j}(a_i) \neq 0$$

If not then,

$$a_i = \sum_{j=1}^{i-1} P_{b_j}(a_i) = \sum_{j=1}^{i-1} (a_i, b_j)\, b_j \in Span\{b_1, b_2, ..., b_{i-1}\} = Span\{a_1, a_2, ..., a_{i-1}\}$$

This is a contradiction to the fact that A is a linearly independent set. Hence,

$$a_i - \sum_{j=1}^{i-1} P_{b_j}(a_i) \neq 0 \implies \|\tilde{b}_i\| \neq 0 \implies \text{ each } b_i \text{ is well defined}$$

From definition we get $\forall\ i \in \{1, 2, ..., n\}$,

$$\|b_i\| = \frac{\|\tilde{b}_i\|}{\|\tilde{b}_i\|} = 1$$

Hence it remains to show that $(b_i, b_j) = 0 \ \forall \ i, j \in \{1, 2, ..., n\}$ where $i < j$.
Claim : $\forall \ i \in \{1, 2, ..., n-1\}$ if $\{b_1, b_2, ..., b_i\}$ is an orthonormal set then,

$$(b_k, b_{i+1}) = 0 \ \forall \ 1 \leq k \leq i$$

$$\begin{aligned}\left(b_k, \tilde{b}_{i+1}\right) &= \left(b_k, a_{i+1} - \sum_{j=1}^{i} P_{b_j}\left(a_{i+1}\right)\right) = \left(b_k, a_{i+1}\right) - \left(b_k, \sum_{j=1}^{i} P_{b_j}\left(a_{i+1}\right)\right) \\ &= \left(b_k, a_{i+1}\right) - \sum_{j=1}^{i} \left(b_k, P_{b_j}\left(a_{i+1}\right)\right) = \left(b_k, a_{i+1}\right) - \sum_{j=1}^{i} \left(b_k, \left(b_j, a_{i+1}\right) b_j\right) \\ &= \left(b_k, a_{i+1}\right) - \sum_{j=1}^{i} \left(b_j, a_{i+1}\right)\left(b_k, b_j\right)\end{aligned}$$

Since $\{b_1, b_2, ..., b_i\}$ is an orthonormal set, then

$$\left(b_k, \tilde{b}_{i+1}\right) = \left(b_k, a_{i+1}\right) - \left(b_k, a_{i+1}\right) = 0 \implies \left(b_k, b_{i+1}\right) = \frac{\left(b_k, \tilde{b}_{i+1}\right)}{\|\tilde{b}_{i+1}\|} = 0$$

Using Lemma 1.1.7, B is a linearly independent set. Since $\dim(V) = n = |B|$ and B is a linearly independent set, B forms a basis of V. Hence, we have shown the existence of an orthonormal basis for V. □

Lemma 1.1.9. Any finite dimensional inner product space over $\mathbb{C}$ with dimension n is isomorphic to the vector space $\mathbb{C}^n$ over $\mathbb{C}$.

Proof. Let V be any finite dimensional inner product space over $\mathbb{C}$. Let $\dim(V) = n < \infty$. To establish an isomorphism between V and $\mathbb{C}^n$, it is enough to provide a linear transformation that maps the basis vectors of V to basis vectors of $\mathbb{C}^n$ bijectively. Consider the standard orthonormal basis $E = \{e_1 = \begin{bmatrix}1 & 0 & . & . & 0\end{bmatrix}^T$, $e_2 = \begin{bmatrix}0 & 1 & . & . & 0\end{bmatrix}^T, ..., e_n = \begin{bmatrix}0 & 0 & . & . & 1\end{bmatrix}^T\}$ of $\mathbb{C}^n$ over $\mathbb{C}$. Since $\dim(V) = n$, there exists a basis $A = \{a_1, a_2, ..., a_n\}$ of V. Consider the linear transformation $T : V \to \mathbb{C}^n$ defined as follows,

$$T\left(a_i\right) = e_i$$

Note that if a linear transformation is defined over each basis vector of V then the linear transformation is well defined $\forall\ x \in V$. It remains to show that T is a bijection in order to establish the isomorphism.

Claim : If $T(x) = 0$ then $x = 0$

From linear algebra it is already known that the above claim implies that T is injective. Since $\dim(V) = \dim(\mathbb{C}^n) = n$ we get that if T is injective then T is surjective. Let's prove the above claim. $\forall\ x \in V$ since A is a basis of V there exist unique $\alpha_i \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

Applying the linear transformation T we get,

$$T(x) = 0 \implies T\left(\sum_{i=1}^{n} \alpha_i a_i\right) = \sum_{i=1}^{n} \alpha_i T(a_i) = \sum_{i=1}^{n} \alpha_i e_i = 0$$

Since E is a linearly independent set we get,

$$\alpha_i = 0\ \forall\ i \in \{1, 2, ..., n\} \implies x = 0$$

$\square$

As a concluding remark, we can see that if we limit our framework to orthonormal basis, any finite dimensional inner product space over $\mathbb{C}$ with dimension n can be identified with $\mathbb{C}^n$ over $\mathbb{C}$ with standard dot product as the inner product.

Chapter 2

Tensor products

2.1 1-fold tensor product spaces - dual spaces

2.1.1 Linear functions

Definition 2.1.1. Let V be a vector space over field $\mathbb{C}$ with inner product $()$: $V \times V \to \mathbb{C}$. A function $u : V \to \mathbb{C}$ is called linear if $\forall\ x, y \in V\ \forall\ \alpha \in \mathbb{C}$,

$$u\,(x+y) = u\,(x) + u\,(y)$$

$$u\,(\alpha \cdot x) = \alpha \cdot u\,(x)$$

Let set $S = \{u : V \to \mathbb{C} \mid u \text{ is linear}\}$. Define addition and scalar multiplication on the set S as follows, $\forall\ u, v \in S$, $\forall\ x \in V$, $\forall\ \alpha \in \mathbb{C}$,

$$[u+v]\,(x) = u\,(x) + v\,(x)$$

$$[\alpha \cdot u]\,(x) = \alpha \cdot u\,(x)$$

Lemma 2.1.1. *S is closed under addition and scalar multiplication*

Proof. **Claim 1 :** $\forall\ u, v \in S$, $[u+v] \in S$

1. $\forall\ x, y \in V$,

$$\begin{aligned}[u+v]\,(x+y) &= u\,(x+y) + v\,(x+y) \\ &= u\,(x) + u\,(y) + v\,(x) + v\,(y) = [u+v]\,(x) + [u+v]\,(y)\end{aligned}$$

2. $\forall\ x \in V, \forall\ \alpha \in \mathbb{C}$,

$$\begin{aligned}[u+v](\alpha x) &= u(\alpha x) + v(\alpha x) \\ &= \alpha u(x) + \alpha v(x) = \alpha[u+v](x)\end{aligned}$$

$$\implies [u+v] \text{ is linear } \implies [u+v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\ u \in V$, $\alpha \in \mathbb{C}$, $[\alpha u] \in S$,

1. $\forall\ x, y \in V$,

$$\begin{aligned}[\alpha u](x+y) &= \alpha u(x+y) \\ &= \alpha u(x) + \alpha u(y) = [\alpha u](x) + [\alpha u](y)\end{aligned}$$

2. $\forall\ x \in V$, $\beta \in \mathbb{C}$,

$$\begin{aligned}[\alpha u](\beta x) &= \alpha u(\beta x) \\ &= \beta\alpha u(x) = \beta[\alpha u](x)\end{aligned}$$

$$\implies [\alpha u] \text{ is linear } \implies [\alpha u] \in S \implies S \text{ is closed under scalar multiplication}$$

□

A linear function $u \in S$ is called a **1-tensor** or a linear map on V. It is easy to verify that S is a vector space over field $\mathbb{C}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to reader). The vectorspace of all 1-tensors is defined to be the 1−fold tensor product space of V denoted by $\mathcal{L}(V \to \mathbb{C})$ or V^*. In addition, 1−fold tensor product space is also called dual space of V.

2.1.2 1-tensors using inner product

Definition 2.1.2. Let V be a finite dimensional inner product space over field $\mathbb{C}$ with $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. $\forall$ $i \in \{1, 2, ..., n\}$ Define $a_i^* : V \to \mathbb{C}$ as follows $\forall\ x \in V$,

$$\boxed{a_i^*(x) = (a_i, x)}$$

Remark :

1. a_i^* is linear $\forall\ i$ and it follows from the definition of inner product 1.1.1

Illustration :
Consider $V = \mathbb{C}^2$ over $\mathbb{C}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{bmatrix}^T, a_2 = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{bmatrix}^T\}$. Let $E = \{e_1 = \begin{bmatrix}1 & 0\end{bmatrix}^T, e_2 = \begin{bmatrix}0 & 1\end{bmatrix}^T\}$ denote the standard orthonormal basis of $\mathbb{C}^2$. Verify that A forms an orthonormal basis of V. From definition 3.1.2 $a_i^* : \mathbb{C}^2 \to \mathbb{C}$ is defined as follows $\forall\ x \in V$,

$$a_i^*(x) = (a_i, x) = \overline{{}^A a_i} \odot {}^A x \text{ where } 1 \leq i \leq 2$$

Note that ${}^A a_i = e_i$

$$\implies a_i^*(x) = \bar{e}_i \odot {}^A x = {}^A x[i]$$

1. $\forall\ x, y \in V$,

$$\begin{aligned} a_i^*(x+y) &= {}^A(x+y)[i] \\ &= {}^A x[i] + {}^A y[i] = a_i^*(x) + a_i^*(y) \end{aligned}$$

2. $\forall\ x \in V\ \forall\ \alpha \in \mathbb{C}$,

$$\begin{aligned} a_i^*(\alpha x) &= {}^A(\alpha x)[i] \\ &= \alpha \cdot {}^A x[i] = \alpha \cdot a_i^*(x) \end{aligned}$$

$$\implies a_i^* \text{ is linear}$$

The linearity of coordinates of vectors (${}^A(x+y) = {}^A x + {}^A y$ and ${}^A(\alpha x) = \alpha \cdot {}^A x$ can be easily verified by expressing vectors x and y in terms of basis A.

2.1.3 Basis of $1-$fold tensor product spaces

Let V be a finite dimensional inner product space over field $\mathbb{C}$ with $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$. From remark 2.1.2 we get that $\forall\ i \in \{1, 2, ..., n\}$ a_i^* is linear.

Lemma 2.1.2. $\forall\ x \in V$,

$$x = \sum_{i=1}^{n} a_i^* (x)\, a_i$$

Proof. Follows directly from lemma 1.1.1 and the definition of a_i^*. Observe that $a_i^* (x)$ gives the i-th coordinate of vector x with respect to basis A. □

Theorem 2.1.3. A^* is a basis for vector space $\mathcal{L}(V \to \mathbb{C})$.

Proof. **Span :** From the above lemma we have that $\forall\ x \in V$,

$$x = \sum_{i=1}^{n} a_i^* (x)\, a_i$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{C})$ since u is linear we get,

$$u(x) = u\left(\sum_{i=1}^{n} a_i^* (x)\, a_i\right) = \sum_{i=1}^{n} u(a_i)\, a_i^* (x)$$

$$\implies A^* \text{ spans } \mathcal{L}(V \to \mathbb{C})$$

Linear Independence : Let $\alpha_i \in \mathbb{C}\ \forall\ 1 \leq i \leq n$. Consider,

$$\sum_{i=1}^{n} \alpha_i a_i^* = 0$$

$\forall\ j \in \{1, 2, ..., n\}$ since A is an orthonormal basis we get that,

$$\sum_{i=1}^{n} \alpha_i a_i^* (a_j) = \sum_{i=1}^{n} \alpha_i (a_i, a_j) = \alpha_j = 0$$

$$\implies A^* \text{ is a linearly independent set and a basis of } \mathcal{L}(V \to \mathbb{C})$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$. A^* is defined to be the dual basis of V.

Corollary 2.1.4.

$$\boxed{\dim\left(\mathcal{L}\left(V \to \mathbb{C}\right)\right) = \dim\left(V^*\right) = \dim\left(V\right)}$$

Illustration :

Consider $V = \mathbb{C}^2$ over $\mathbb{C}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Let $A^* = \{a_1^*,\ a_2^*\}$ where each $a_i^* : \mathbb{C}^2 \to \mathbb{C}$ is defined as follows $\forall\ x \in V$,

$$a_i^*\left(x\right) = \left(a_i, x\right) = \overline{{}^{A}a_i} \odot {}^{A}x = {}^{A}x[i] \text{ where } 1 \leq i \leq 2$$

From the above theorem we get that A^* is a basis of $\mathcal{L}\left(\mathbb{C}^2 \to \mathbb{C}\right)$. Consider the following linear function,

$$u\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = x + y \qquad \text{where } \begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{C}^2$$

It is straight forward to verify that u is linear. Since A is a basis of V $\forall \begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{C}^2$ there exist unique α, $\beta \in \mathbb{C}$ such that,

$$\begin{bmatrix} x & y \end{bmatrix}^T = \alpha a_1 + \beta a_2 \implies a_1^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = \alpha \text{ and } a_2^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = \beta$$

Since u is linear we get that,

$$u\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = a_1^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) u\left(a_1\right) + a_2^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) u\left(a_2\right)$$

From the above expression it is quite clear that computing $u\left(a_1\right)$ and $u\left(a_2\right)$ is sufficient to determine the action of u on any $\begin{bmatrix} x & y \end{bmatrix}^T$.

$$u\left(a_1\right) = u\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T\right) = \frac{1+i}{\sqrt{2}} \qquad u\left(a_2\right) = u\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\right) = \frac{1-i}{\sqrt{2}}$$

$$\implies u\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = \frac{1+i}{\sqrt{2}} a_1^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) + \frac{1-i}{\sqrt{2}} a_2^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right)$$

$$\implies u\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = u(a_1)\, a_1^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) + u(a_2)\, a_2^*\left(\begin{bmatrix} x & y \end{bmatrix}^T\right)$$

With this illustration, observe how $\{a_1^*, a_2^*\}$ works as a basis of $\mathcal{L}(\mathbb{C}^2 \to \mathbb{C})$ and also note that the values $u(a_1)$, $u(a_2)$ is sufficient to compute $u\left(\begin{bmatrix} x & y \end{bmatrix}^T\right)$ for any $\begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{C}^2$

Remark :

1. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})$ from theorem 2.1.3 we get that,

$$u = \sum_{i=1}^{n} u(a_i)\, a_i^*$$

From definition 1.1.2 we get coordinates of the vector $u \in \mathcal{L}(V \to \mathbb{C})$ with respect to basis A^* as follows,

$$^{A^*}u = \begin{bmatrix} u(a_1) & u(a_2) & . & . & u(a_n) \end{bmatrix}^T$$

2.1.4 Basis Transformation

Theorem 2.1.5. Let V be a finite dimensional inner product space over field $\mathbb{C}$ with $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$. Theorem 2.1.3 implies that A^* and B^* form bases of $\mathcal{L}(V \to \mathbb{C})$. Let $M \in \mathbb{C}^{n\times n}$ be the transformation matrix from basis A to B i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix} M$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{C})$,

$$\boxed{^{B^*}u = \overline{M} \cdot {}^{A^*}u}$$

Proof. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})$ since A^* and B^* form bases of $\mathcal{L}(V \to \mathbb{C})$ we get,

$$u = \sum_{j=1}^{n} u(a_j)\, a_j^* = \sum_{i=1}^{n} u(b_i)\, b_i^*$$

$$\implies {}^{A^*}u = \begin{bmatrix} u(a_1) & u(a_2) & . & . & . & u(a_n) \end{bmatrix}^T$$

$$ {}^{B^*}u = \begin{bmatrix} u(b_1) & u(b_2) & . & . & . & u(b_n) \end{bmatrix}^T$$

Since M is the transformation matrix from basis A to B we get that M^* is the transformation matrix from B to A i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} M^*$$

$\forall\ i \in \{1, 2, ..., n\}$,

$$b_i = \sum_{j=1}^{n} M^*_{ji} a_j = \sum_{j=1}^{n} \overline{M}_{ij} \cdot a_j$$

Since u is linear we get,

$$u(b_i) = u\left(\sum_{j=1}^{n} \overline{M}_{ij} \cdot a_j\right) = \sum_{j=1}^{n} \overline{M}_{ij} \cdot u(a_j)$$

$$\implies u(b_i) = \begin{bmatrix} \overline{M}_{i1} & \overline{M}_{i2} & . & . & \overline{M}_{in} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \\ . \\ . \\ u(a_n) \end{bmatrix}$$

$$\implies \begin{bmatrix} u(b_1) \\ u(b_2) \\ . \\ . \\ u(b_n) \end{bmatrix} = \begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} & . & . & \overline{M}_{1n} \\ \overline{M}_{21} & \overline{M}_{22} & . & . & \overline{M}_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \overline{M}_{n1} & \overline{M}_{n2} & . & . & \overline{M}_{nn} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \\ . \\ . \\ u(a_n) \end{bmatrix}$$

$$\implies {}^{B^*}u = \overline{M} \cdot {}^{A^*}u$$

$\square$

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal basis of V. Then, $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$ form bases of $\mathcal{L}(V \to \mathbb{C})$. Let $M \in \mathbb{C}^{n\times n}$ be the transformation matrix from basis A to B. Then, $\overline{M}$ is the basis transformation matrix from basis A^* to basis B^* and $\left(\overline{M}\right)^{-1} = M^T$ is the basis transformation matrix from B to A. More concretely $\forall\ u \in \mathcal{L}(V \to \mathbb{C})$,

$$\begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} = \begin{bmatrix} b_1^* & b_2^* & . & . & b_n^* \end{bmatrix} \overline{M} \implies {}^{B^*}u = \overline{M} \cdot {}^{A*}u$$

$$\implies \boxed{{}^{A^*}u = M^T \cdot {}^{B}u} \implies \begin{bmatrix} b_1^* & b_2^* & . & . & b_n^* \end{bmatrix} = \begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} M^T$$

Illustration :

Consider $V = \mathbb{C}^2$ over $\mathbb{C}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Let $A^* = \{a_1^*, a_2^*\}$ where each $a_i^* : \mathbb{C}^2 \to \mathbb{C}$ is defined as follows $\forall\ x \in V$,

$$a_i^*(x) = (a_i, x) = {}^{A}a_i \odot {}^{A}x = {}^{A}x[i] \text{ where } 1 \le i \le 2$$

Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$. Let $B^* = \{b_1^*, b_2^*\}$ where each $b_i^* : \mathbb{C}^2 \to \mathbb{C}$ is defined as follows $\forall\ x \in V$,

$$b_i^*(x) = (b_i, x) = {}^{B}b_i \odot {}^{B}x = {}^{B}x[i] \text{ where } 1 \le i \le 2$$

Verify that both A and B are orthonormal bases of $\mathbb{C}^2$. From theorem 2.1.3 we get that A^* and B^* form bases of $\mathcal{L}(V \to \mathbb{C})$.

Computing the basis transformation matrix from A^* to B^*

$$a_1 = \frac{1}{\sqrt{2}}b_1 - \frac{i}{\sqrt{2}}b_2 \qquad a_2 = \frac{1}{\sqrt{2}}b_1 + \frac{i}{\sqrt{2}}b_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to B of $\mathbb{C}^2$ which implies that M^* is the transformation matrix from basis B to A i.e,

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$$

$$\implies b_1 = \frac{1}{\sqrt{2}} a_1 + \frac{1}{\sqrt{2}} a_2 \qquad\qquad b_2 = \frac{i}{\sqrt{2}} a_1 - \frac{i}{\sqrt{2}} a_2$$

$\forall\ u \in \mathcal{L}(\mathbb{C}^2 \to \mathbb{C})$ since A^* and B^* form bases of $\mathcal{L}(\mathbb{C}^2 \to \mathbb{C})$ we get,

$$u = u(a_1)\, a_1^* + u(a_2)\, a_2^* = u(b_1)\, b_1^* + u(b_2)\, b_2^*$$

Since u is linear we get that,

$$u(b_1) = \frac{1}{\sqrt{2}} u(a_1) + \frac{1}{\sqrt{2}} u(a_2) \qquad u(b_2) = \frac{i}{\sqrt{2}} u(a_1) - \frac{i}{\sqrt{2}} u(a_2)$$

$$\implies \begin{bmatrix} u(b_1) \\ u(b_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \end{bmatrix} \implies {}^{B^*}u = \overline{M} \cdot {}^{A^*}u$$

2.1.5 Invariance of computation of $1-$tensors under any orthonormal basis transformations

Theorem 2.1.6. Let V be a finite dimensional inner product space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$. Theorem 2.1.3 implies that A^* forms a basis of $\mathcal{L}(V \to \mathbb{C})$. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})\ \forall\ x \in V$,

$$\boxed{u(x) = \sum_{r=1}^{n} {}^{A^*}u[r] \cdot {}^{A}x[r] = \overline{{}^{A^*}u} \odot {}^{A}x}$$

Proof. $\forall\ x \in V$ since A is a basis of V there exist unique $\alpha_r \in \mathbb{C}$ such that,

$$x = \sum_{r=1}^{n} \alpha_r a_r$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{C})$ since u is linear we get,

$$u(x) = u\left(\sum_{r=1}^{n} \alpha_r a_r\right) = \sum_{r=1}^{n} \alpha_r u(a_r)$$

Since A^* is a basis of $\mathcal{L}(V \to \mathbb{C})$ and A is a basis of V we get,

$$u(a_r) = {}^{A^*}u[r] \qquad\qquad \alpha_r = {}^{A}x[r]$$

$$\implies u(x) = \sum_{r=1}^{n} {}^{A^*}u[r] \cdot {}^{A}x[r] = \overline{{}^{A^*}u} \odot {}^{A}x$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ be any two orthonormal bases of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$. Note that both A^* and B^* form bases of $\mathcal{L}(V \to \mathbb{C})$. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})\ \forall\ x \in V$,

$$u(x) = \overline{{}^{A^*}u} \odot {}^{A}x = \overline{{}^{B^*}u} \odot {}^{B}x$$

2. It is easy to observe that $\forall\ x \in V\ u(x)$ can be determined by the euclidean dot product of ${}^{A^*}u$ and ${}^{A}x$. Hence if we fix computations with respect to an orthonormal basis A we can identify u with ${}^{A^*}u$

3. Let $\dim(V) = n$. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})$,

$$\left({}^{A^*}u\right)^T = \begin{bmatrix} u(a_1) & u(a_2) & . & . & u(a_n) \end{bmatrix}^T$$

is a $1 \times n$ row vector in $\mathbb{C}^n$. Hence it is easy to show that $1-$fold tensor product space i.e, $\mathcal{L}(V \to \mathbb{C})$ is isomorphic to $\mathbb{C}^{1\times n} \cong \mathbb{C}^n$. (It is straightforward to verify and left to reader. For proof technique refer lemma 1.1.9)

4.
$$u(x) = \overline{{}^{B^*}u} \odot {}^{B}x = \left(M \cdot \overline{{}^{A^*}u}\right) \odot \left(M \cdot {}^{A}x\right) = \overline{{}^{A^*}u} \odot {}^{A}x$$

$$\implies \boxed{u(x) = \overline{{}^{B^*}u} \odot {}^{B}x = \overline{{}^{A^*}u} \odot {}^{A}x}$$

Illustration :

Consider $V = \mathbb{C}^2$ over $\mathbb{C}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Let $A^* = \{a_1^*, a_2^*\}$ where each $a_i^* : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows $\forall\ x \in V$,

$$a_i^*(x) = (a_i, x) = {}^{A}a_i \odot {}^{A}x = {}^{A}x[i] \text{ where } 1 \leq i \leq 2$$

Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$. Let $B^* = \{b_1^*, b_2^*\}$ where each $b_i^* : \mathbb{C}^2 \to \mathbb{C}$ is defined as follows $\forall\ x \in V$,

$$b_i^*(x) = (b_i, x) = {}^{B}b_i \odot {}^{B}x = {}^{B}x[i] \text{ where } 1 \leq i \leq 2$$

Verify that both A and B form orthonormal bases of $\mathbb{C}^2$. From Theorem 2.1.3 we get A^* and B^* form bases of $\mathcal{L}(V \to \mathbb{C})$. In the previous illustration we have already shown that the basis transformation matrix M from A to B i.e,

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

We have also seen that $\forall\ x \in \mathbb{C}^2\ \forall\ u \in \mathcal{L}(\mathbb{C}^2 \to \mathbb{C})$,

$${}^{B}x = M \cdot {}^{A}x \qquad\qquad {}^{B^*}u = \overline{M} \cdot {}^{A^*}u$$

Since B is a basis of $\mathbb{C}^2$ we get,

$$x = {}^{B}x[1]b_1 + {}^{B}x[2]b_2$$

Since u is linear we get that,

$$u(x) = u\left({}^{B}x[1]b_1 + {}^{B}x[2]b_2\right) = {}^{B}x[1]u(b_1) + {}^{B}x[2]u(b_2) = \overline{{}^{B^*}u} \odot {}^{B}x$$

$$\implies u(x) = \left(M \cdot \overline{{}^{A^*}u}\right) \odot \left(M \cdot {}^{A}x\right) = \overline{{}^{A^*}u} \odot {}^{A}x$$

$$\implies u(x) = \overline{{}^{B^*}u} \odot {}^{B}x = \overline{{}^{A^*}u} \odot {}^{A}x$$

2.1.6 Inner products on $1-$fold tensor product spaces

In this section we define a function $() : V^* \times V^* \to \mathbb{C}$ and prove that this function is an inner product which shows the existence of inner product on dual space V^* i.e, V^* is an inner product space.

Definition 2.1.3. Let V be a finite dimensional inner product space where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V and $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ be the corresponding dual basis. $\forall\, u, v \in \mathcal{L}(V \to \mathbb{C})$ we define the following function $() : V^* \times V^* \to \mathbb{C}$ as follows,

$$\boxed{(u, v) = {}^{A^*}u \odot {}^{A^*}v}$$

Lemma 2.1.7. $\forall\, u, v \in \mathcal{L}(V \to \mathbb{C})$,

$$(u, v) = {}^{A^*}u \odot {}^{A^*}v \text{ is an inner product.}$$

Proof. **Linearity :** $\forall\, u, v, w \in \mathcal{L}(V \to \mathbb{C})\ \forall\, \alpha \in \mathbb{C}$,

$$\begin{aligned}(u, v + w) &= {}^{A^*}u \odot {}^{A^*}(v + w) = {}^{A^*}u \odot \left({}^{A^*}v + {}^{A^*}w\right) = {}^{A^*}u \odot {}^{A^*}v + {}^{A^*}u \odot {}^{A^*}w \\ &= (u, v) + (u, w)\end{aligned}$$

$$(u, \alpha v) = {}^{A^*}u \odot {}^{A^*}(\alpha v) = {}^{A^*}u \odot \left(\alpha \cdot {}^{A^*}v\right) = \alpha \cdot {}^{A^*}u \odot {}^{A^*}v = \alpha\,(u, v)$$

Conjugate Symmetry : $\forall\, u, v \in \mathcal{L}(V \to \mathbb{C})$,

$$\begin{aligned}(u, v) &= {}^{A^*}u \odot {}^{A^*}v = \sum_{r=1}^{n} \overline{{}^{A^*}u[r]} \cdot {}^{A^*}v[r] = \overline{\sum_{r=1}^{n} \overline{{}^{A^*}v[r]} \cdot {}^{A^*}u[r]} \\ &= \overline{{}^{A^*}v \odot {}^{A^*}u} = \overline{(v, u)}\end{aligned}$$

Positive Definiteness : $\forall\, u \in \mathcal{L}(V \to \mathbb{C})$,

$$(u, u) = 0 \iff {}^{A^*}u \odot {}^{A^*}u = 0 \iff \sum_{r=1}^{n} \left|{}^{A^*}u[r]\right|^2 = 0 \iff {}^{A^*}u = 0 \iff u = 0$$

□

Lemma 2.1.8. $\forall\, u, v \in \mathcal{L}(V \to \mathbb{C})$,

$$(u, v) = {}^{A^*}u \odot {}^{A^*}v \text{ is well-defined.}$$

Proof. Let $B = \{b_1, b_2, ..., b_n\}$ be any another orthonormal basis of V. Let $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$ be the corresponding dual basis. To claim (u, v) is well-defined it is enough to show that

$$\left(\sum_{i=1}^{n} u(a_i)\, a_i^*, \sum_{j=1}^{n} v(a_j)\, a_j^*\right) = \left(\sum_{p=1}^{n} u(b_p)\, b_p^*, \sum_{q=1}^{n} v(b_q)\, b_q^*\right)$$

$$\left(\sum_{i=1}^{n} u(a_i)\, a_i^*, \sum_{j=1}^{n} v(a_j)\, a_j^*\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{u(a_i)} u(a_j)\left(a_i^*, a_j^*\right)$$

Let $M \in \mathbb{C}^{n \times n}$ be the transformation matrix from basis A to B. Theorem 2.1.5 implies that

$$\begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} = \begin{bmatrix} b_1^* & b_2^* & . & . & b_n^* \end{bmatrix} \overline{M}$$

$$\implies a_i^* = \sum_{p=1}^{n} \overline{M}_{pi} b_p^* \qquad a_j^* = \sum_{q=1}^{n} \overline{M}_{qj} b_q^*$$

$$\begin{aligned} \left(\sum_{i=1}^{n} u(a_i)\, a_i^*, \sum_{j=1}^{n} v(a_j)\, a_j^*\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} M_{pi}\overline{M}_{qj}\overline{u(a_i)} u(a_j)\left(a_i^*, a_j^*\right) \\ &= \sum_{p=1}^{n}\sum_{q=1}^{n} \overline{\left(\sum_{i=1}^{n} \overline{M}_{pi} u(a_i)\right)} \left(\sum_{j=1}^{n} \overline{M}_{qj} u(a_j)\right)\left(b_p^*, b_q^*\right) \end{aligned}$$

Theorem 2.1.5 implies that,

$${}^{B^*}u = \overline{M} \cdot {}^{A^*}u \implies u(b_p) = \sum_{i=1}^{n} \overline{M}_{pi} u(a_i) \text{ and } u(b_q) = \sum_{j=1}^{n} \overline{M}_{qj} u(a_j)$$

$$\implies \left(\sum_{i=1}^{n} u(a_i)\, a_i^*, \sum_{j=1}^{n} v(a_j)\, a_j^*\right) = \left(\sum_{p=1}^{n} u(b_p)\, b_p^*, \sum_{q=1}^{n} v(b_q)\, b_q^*\right)$$

□

We end this section showing how inner products can be used in giving an alternate proof for the linear independence of dual basis. $\forall\ i, j \in \{1, 2, ..., n\}$,

$$\begin{aligned}\left(a_i^*, a_j^*\right) = {}^{A^*}a_i^* \odot {}^{A^*}a_j^* = e_i \odot e_j &= 1 \quad \text{if } i = j \\ &= 0 \quad \text{if } i \neq j\end{aligned}$$

From lemma 1.1.7 it is straight forward to verify linear independence of dual basis.

2.1.7 Linear operators on 1-fold tensor product spaces

Definition 2.1.4. Let $\mathcal{L}\left(V^*\right)$ denote the set of all linear operators over the dual space $\mathcal{L}\left(V \rightarrow \mathbb{C}\right) = V^*$. Define addition and scalar multiplication on the set $\mathcal{L}\left(V^*\right)$ as follows $\forall\ T, W \in \mathcal{L}\left(V^*\right)\ \forall\ x \in V^*\ \forall\ \alpha \in \mathbb{C}$,

$$[T + W]x = T\left(x\right) + W\left(x\right)$$

$$[\alpha T]x = \alpha \cdot T\left(x\right)$$

Remark :

1. It is straight forward to verify that $\mathcal{L}\left(V^*\right)$ is a vector space over $\mathbb{C}$ and is left to reader (refer lemma 2.1.1).
2. Note that $\mathcal{L}\left(V^*\right)$ is also called tensor product space of operators on the dual space V^*.

Next we shall find a basis of $\mathcal{L}\left(V^*\right)$.

Definition 2.1.5. Let V be a finite dimensional inner product space over field $\mathbb{C}$ where $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ be the corresponding dual basis. Define $T^* = \{T_{ij} \mid 1 \leq i, j \leq n\}$ where $\forall\ i, j, k \in \{1, 2, ..., n\}$ $T_{ij} \in \mathcal{L}(V^*)$ is defined as follows,

$$\begin{aligned}T_{ij}\left(a_k^*\right) &= a_j^* \qquad \text{if } k = i \\ &= 0 \qquad \text{if } k \neq i\end{aligned}$$

Remark :

1. Note that each $T_{ij} \in T^*$ is well-defined since A^* is a basis of V^* and T_{ij} is defined on each basis element of V^* (Recall that in order to define a linear operator it is enough to define the operator on a basis of the vector space).

2. Note that by definition each T_{ij} is linear.

Theorem 2.1.9. *T^* forms a basis of $\mathcal{L}(V^*)$.*

Proof. **Span :** $\forall\, u \in V^*$ since A^* is a basis of V^* there exist unique $\alpha_i \in \mathbb{C}$ such that,

$$u = \sum_{i=1}^{n} \alpha_i a_i^*$$

$\forall\, W \in \mathcal{L}(V^*)$ since W is a linear operator we get,

$$W(u) = \sum_{i=1}^{n} \alpha_i W(a_i^*)$$

Since W is an operator $\forall\, i \in \{1, 2, ..., n\}$ there exist $\beta_{ij} \in \mathbb{C}$ such that,

$$W(a_i^*) = \sum_{j=1}^{n} \beta_{ij} a_j^*$$

$$\implies W(u) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \alpha_i a_j^*$$

Note that $\forall\, i, j \in \{1, 2, ..., n\}$ since T_{ij} is linear,

$$T_{ij}(u) = \sum_{k=1}^{n} \alpha_k T_{ij}(a_k^*) = \alpha_i a_j^*$$

$$\implies W(u) = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} T_{ij}(u) \implies W = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} T_{ij}$$

$$\implies T^* \text{ spans } \mathcal{L}(V^*)$$

Linear Independence : $\forall\ i, j \in \{1, 2, ..., n\}$. Consider,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} T_{ij} = 0$$

$\forall\ k \in \{1, 2, ..., n\}$ applying a_k^* we get,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} T_{ij}\left(a_k^*\right) = 0 \implies \sum_{j=1}^{n} \alpha_{kj} a_j^* = 0$$

Since A^* is a basis of V^* we get that,

$$\alpha_{kj} = 0 \qquad \forall\ j \in \{1, 2, ..., n\}$$

$$\implies T^* \text{ is a linearly independent set and forms a basis of } \mathcal{L}(V^*)$$

$\square$

Corollary 2.1.10.

$$\boxed{\dim\left(\mathcal{L}\left(V^*\right)\right) = \left(\dim\left(V^*\right)\right)^2}$$

It will be evident in the next sections on how the basis T^* is used to construct the basis of $2, 3, k$-fold tensor product space of operators.

2.2 2-fold tensor product spaces

2.2.1 Bi-linear functions

Definition 2.2.1. Let V, W be vector spaces over field $\mathbb{C}$ with inner products $()_1 : V \times V \to \mathbb{C}$ and $()_2 : W \times W \to \mathbb{C}$. Note that subscripts $()_1$ and $()_2$ will be dropped if the context is clear. A function $u : V \times W \to \mathbb{C}$ is called bi-linear if the following holds,

1. $\forall\ x, y \in V\ \forall\ z \in W$,

$$u\left(x+y, z\right) = u\left(x, z\right) + u\left(y, z\right)$$

2. $\forall\ x \in V\ \forall\ y, z \in W$,

$$u\left(x, y+z\right) = u\left(x, y\right) + u\left(x, z\right)$$

3. $\forall\ x \in V\ \forall\ y \in W\ \forall\ \alpha \in \mathbb{C}$,

$$u\left(\alpha x, y\right) = \alpha u\left(x, y\right) = u\left(x, \alpha y\right)$$

Let set $S = \{u : V \times W \to \mathbb{C} \mid u \text{ is bi-linear}\}$. Define addition and multiplication on the set S as follows, $\forall\ u, v \in S\ \forall\ x \in V\ \forall\ y \in W\ \forall\ \alpha \in \mathbb{C}$,

$$[u+v]\left(x, y\right) = u\left(x, y\right) + v\left(x, y\right)$$
$$[\alpha u]\left(x, y\right) = \alpha u\left(x, y\right)$$

Lemma 2.2.1. *S is closed under addition and scalar multiplication*

Proof. **Claim 1 :** $\forall\ u, v \in S\ [u+v] \in S$,

1. $\forall\ x, y \in V, \forall\ z \in W$,

$$\begin{aligned}[u+v]\left(x+y, z\right) &= u\left(x+y, z\right) + v\left(x+y, z\right) = u\left(x, z\right) + u\left(y, z\right) + v\left(x, z\right) + v\left(y, z\right)\\ &= [u+v]\left(x, z\right) + [u+v]\left(y, z\right)\end{aligned}$$

2. $\forall\ x \in V\ \forall\ y, z \in W$,

$$\begin{aligned}[u+v](x, y+z) = u(x, y+z) + v(x, y+z) &= u(x,y) + u(x,z) + v(x,y) + v(x,z) \\ &= [u+v](x,y) + [u+v](x,z)\end{aligned}$$

3. $\forall\ x \in V\ \forall\ y \in W\ \forall\ \alpha \in \mathbb{C}$,

$$\begin{aligned}[u+v](\alpha x, y) = u(\alpha x, y) + v(\alpha x, y) &= \alpha u(x,y) + \alpha v(x,y) = \alpha [u+v](x,y) \\ &= u(x, \alpha y) + v(x, \alpha y) = [u+v](x, \alpha y)\end{aligned}$$

$$[u+v] \text{ is bi-linear } \implies [u+v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\ u \in S\ \alpha \in \mathbb{C}\ [\alpha u] \in S$,

1. $\forall\ x, y \in V\ \forall\ z \in W$,

$$[\alpha u](x+y, z) = \alpha u(x+y, z) = \alpha u(x,z) + \alpha u(y,z) = [\alpha u](x,z) + [\alpha u](y,z)$$

2. $\forall\ x \in V\ \forall\ y, z \in W$,

$$[\alpha u](x, y+z) = \alpha u(x, y+z) = \alpha u(x,y) + \alpha u(x,z) = [\alpha u](x,y) + [\alpha u](x,z)$$

3. $\forall\ x \in V\ \forall\ y \in W\ \forall\ \beta \in \mathbb{C}$,

$$[\alpha u](\beta x, y) = \alpha u(\beta x, y) = \alpha\beta u(x,y) = \beta [\alpha u](x,y) = \alpha u(x, \beta y) = [\alpha u](x, \beta y)$$

$$[\alpha u] \text{ is bi-linear } \implies [\alpha u] \in S \implies S \text{ is closed under scalar multiplication}$$

□

A bi-linear function $u \in S$ is called a **2-tensor** or a bi-linear map on $V \times W$. It is easy to verify that S is a vector space over field $\mathbb{C}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all 2-tensors is defined to be the 2-fold tensor product space of V and W denoted by $\mathcal{L}(V \times W \to \mathbb{C})$ or $V \otimes W$.

2.2.2 Tensor products on vector spaces V and W

Definition 2.2.2. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V) = n$ and $dim\,(W) = m$. $\forall\ u \in \mathcal{L}\,(V \to \mathbb{C})\ \forall\ v \in \mathcal{L}\,(W \to \mathbb{C})$ Define the tensor product of u and v as a function $[u \otimes v] : V \times W \to \mathbb{C}$ as follows $\forall\ x \in V\ \forall\ y \in W$,

$$\boxed{[u \otimes v]\,(x, y) = u\,(x) \cdot v\,(y)}$$

Remark :

1. $\forall\ u, v \in \mathcal{L}\,(V \to \mathbb{C})$ notice that $u \otimes v \neq v \otimes u$ in general.

Lemma 2.2.2. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$. $\forall\ u \in \mathcal{L}\,(V \to \mathbb{C})\ \forall\ v \in \mathcal{L}\,(W \to \mathbb{C})$, $[u \otimes v]$ is bi-linear.

Proof. 1. $\forall\ x, y \in V\ \forall\ z \in W$,

$$\begin{aligned}[u \otimes v]\,(x + y, z) &= u\,(x + y) \cdot v\,(z) = u\,(x) \cdot v\,(z) + u\,(y) \cdot v\,(z) \\ &= [u \otimes v]\,(x, z) + [u \otimes v]\,(y, z)\end{aligned}$$

2. $\forall\ x \in V\ \forall\ y, z \in W$,

$$\begin{aligned}[u \otimes v]\,(x, y + z) &= u\,(x) \cdot v\,(y + z) = u\,(x) \cdot v\,(y) + u\,(x) \cdot v\,(z) \\ &= [u \otimes v]\,(x, y) + [u \otimes v]\,(x, z)\end{aligned}$$

3. $\forall\ x \in V\ \forall\ y \in W\ \forall\ \alpha \in \mathbb{C}$,

$$\begin{aligned}[u \otimes v]\,(\alpha x, y) &= u\,(\alpha x) \cdot v\,(y) = u\,(x) \cdot v\,(\alpha y) = [u \otimes v]\,(x, \alpha y) \\ &= \alpha u\,(x) \cdot v\,(y) = \alpha\,[u \otimes v]\,(x, y)\end{aligned}$$

$$\implies [u \otimes v] \text{ is bi-linear } \implies [u \otimes v] \in \mathcal{L}\,(V \times W \to \mathbb{C})$$

□

Lemma 2.2.3. Let V, W be any two vector spaces over field $\mathbb{C}$.

1. $\forall\ u, v \in \mathcal{L}(V \to \mathbb{C})\ \forall\ w \in \mathcal{L}(W \to \mathbb{C})$,

$$[u + v] \otimes w = u \otimes w + v \otimes w$$

2. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})\ \forall\ v, w \in \mathcal{L}(W \to \mathbb{C})$,

$$u \otimes [v + w] = u \otimes v + u \otimes w$$

3. $\forall\ u \in \mathcal{L}(V \to \mathbb{C})\ \forall\ v \in \mathcal{L}(W \to \mathbb{C})\ \forall\ \alpha \in \mathbb{C}$,

$$[\alpha u] \otimes v = u \otimes [\alpha v] = \alpha\,[u \otimes v]$$

Proof. $\forall\ x \in V\ \forall\ y \in W$,

1. $\forall\ u, v \in V\ \forall\ w \in W$,

$$\begin{aligned}[][[u + v] \otimes w](x, y) &= [u + v](x) \cdot w(y) = u(x) \cdot w(y) + v(x) \cdot w(y) \\ &= [u \otimes w](x, y) + [v \otimes w](x, y)\end{aligned}$$

2. $\forall\ u \in V\ \forall\ v, w \in W$,

$$\begin{aligned}[][u \otimes [v + w]](x, y) &= u(x) \cdot [v + w](y) = u(x) \cdot v(y) + u(x) \cdot w(y) \\ &= [u \otimes v](x, y) + [u \otimes w](x, y)\end{aligned}$$

3. $\forall\ u \in V\ \forall\ v \in W\ \forall\ \alpha \in \mathbb{C}$,

$$\begin{aligned}[][[\alpha u] \otimes v](x, y) &= [\alpha u](x) \cdot v(y) = \alpha u(x) \cdot v(y) = \alpha\,[u \otimes v](x, y) \\ &= u(x) \cdot [\alpha v](y) = [u \otimes [\alpha v]](x, y)\end{aligned}$$

□

Remark :

1. $u \otimes v = 0 \iff u = 0$ or $v = 0$. It is straight forward to verify and left to reader.

2. Note that the tensor products don't have unique representations for instance $\forall\ u \in \mathcal{L}(V \to \mathbb{C})\ v \in \mathcal{L}(W \to \mathbb{C})\ \forall\ \alpha \neq 0 \in \mathbb{C}$,

$$u \otimes v = \frac{u}{\alpha} \otimes (\alpha v)$$

Illustration :

Consider $V = \mathbb{C}^2$ over $\mathbb{C}$ and $W = \mathbb{C}^3$ over $\mathbb{C}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$ and $B = \{b_1 = \begin{bmatrix} \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \end{bmatrix}^T, b_2 = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{bmatrix}^T, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T\}$. Verify that A and B form orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let $A^* = \{a_1^*, a_2^*\}$ and $B^* = \{b_1^*, b_2^*, b_3^*\}$. Recall that A^* and B^* are dual bases of V^* and W^* respectively. Theorem 2.1.6 implies that $\forall\ u \in V^*\ \forall\ v \in W^*\ \forall\ x \in \mathbb{C}^2\ \forall\ y \in \mathbb{C}^3$,

$$u(x) = \overline{{}^{A^*}u} \odot {}^{A}x \qquad\qquad v(y) = \overline{{}^{B^*}v} \odot {}^{B}y$$

Let $u = \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$, $v = \begin{bmatrix} b_1^* & b_2^* & b_3^* \end{bmatrix} \begin{bmatrix} 3 \\ 2i \\ 1 \end{bmatrix}$.

$\forall\ x \in \mathbb{C}^2$, $\forall\ y \in \mathbb{C}^3$,

$$[u \otimes v](x, y) = u(x) \cdot v(y) = \left(\overline{{}^{A^*}u} \odot {}^{A}x\right) \cdot \left(\overline{{}^{B^*}v} \odot {}^{B}y\right) = {}^{A}x^T \cdot {}^{A^*}u \cdot {}^{B^*}v^T \cdot {}^{B}y$$

$$\implies [u \otimes v](x, y) = \begin{bmatrix} {}^{A}x[1] & {}^{A}x[2] \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2i \end{bmatrix} \cdot \begin{bmatrix} 3 & 2i & 1 \end{bmatrix} \cdot \begin{bmatrix} {}^{B}y[1] \\ {}^{B}y[2] \\ {}^{B}y[3] \end{bmatrix}$$

$$\implies [u \otimes v](x, y) = \begin{bmatrix} {}^{A}x[1] & {}^{A}x[2] \end{bmatrix} \cdot \begin{bmatrix} 3 & 2i & 1 \\ 6i & -4 & 2i \end{bmatrix} \begin{bmatrix} {}^{B}y[1] \\ {}^{B}y[2] \\ {}^{B}y[3] \end{bmatrix}$$

From the linearity of coordinates of vectors x, y it is straight forward to conclude that $[u \otimes v]$ is bi-linear and all the properties in lemma 2.2.3 hold. Note that computing ${}^{A^*}u \cdot \left({}^{B^*}v\right)^T$ is sufficient to determine the action of $[u \otimes v]$ on any $(x, y) \in \mathbb{C}^2 \times \mathbb{C}^3$.

2.2.3 Basis of 2-fold tensor product spaces

Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V) = n$ and $dim\,(W) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V and $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of W. Define $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Lemma 2.2.2 implies that $\forall\ i \in \{1, 2, ..., n\}$ $\forall\ j \in \{1, 2, ..., m\}$ $a_i^* \otimes b_j^*$ is bi-linear.

Theorem 2.2.4. *$A \otimes B$ is a basis for vector space $\mathcal{L}\,(V \times W \to \mathbb{C})$*

Proof. **Span :**
$\forall\ x \in V$ since A is a basis of V there exist unique $\alpha_i \in \mathbb{C}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

$\forall\ y \in W$ since B is a basis of W there exist unique $\beta_j \in \mathbb{C}$ such that

$$y = \sum_{j=1}^{m} \beta_j b_j$$

$\forall\ u \in \mathcal{L}\,(V \times W \to \mathbb{C})$ since u is bi-linear we get that,

$$u\,(x, y) = u\left(\sum_{i=1}^{n} \alpha_i a_i, \sum_{j=1}^{m} \beta_j b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j u\,(a_i, b_j)$$

$\forall\ i \in \{1, 2, ..., n\}$ $\forall\ j \in \{1, 2, ..., m\}$ from definition 1.1.2 we get

$$\left[a_i^* \otimes b_j^*\right](x, y) = (a_i, x)\,(b_j, y) = {}^{A}x\,[i] \cdot {}^{B}y\,[j] = \alpha_i \beta_j$$

$$\implies u\,(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} u\,(a_i, b_j)\left[a_i^* \otimes b_j^*\right](x, y) \implies u = \sum_{i=1}^{n}\sum_{j=1}^{m} u\,(a_i, b_j)\left[a_i^* \otimes b_j^*\right]$$

$$\implies A \otimes B \text{ spans } \mathcal{L}\,(V \times W \to \mathbb{C})$$

Linear Independence :
Let $\alpha_{ij} \in \mathbb{C}\ \forall\ i \in \{1,2,...,n\}\ \forall\ j \in \{1,2,...,m\}$. Consider,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}\left[a_i^* \otimes b_j^*\right] = 0$$

$\forall\ p \in \{1,2,...,n\}\ \forall\ q \in \{1,2,...,m\}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}\left[a_i^* \otimes b_j^*\right](a_p, b_q) = 0 \implies \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}\left(a_i, a_p\right)\left(b_j, b_q\right) = 0$$

Since A and B are orthonormal bases we get that,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}\left(a_i, a_p\right)\left(b_j, b_q\right) = \alpha_{pq} = 0$$

$$\implies A \otimes B \text{ is a linearly independent set and a basis of } \mathcal{L}\left(V \times W \to \mathbb{C}\right)$$

□

Corollary 2.2.5.

$$\boxed{\dim\left(\mathcal{L}\left(V \times W \to \mathbb{C}\right)\right) = \dim\left(V \otimes W\right) = \dim\left(V\right) \cdot \dim\left(W\right)}$$

Illustration :
Consider $V = \mathbb{C}^2$ over $\mathbb{C}$ and $W = \mathbb{C}^3$ over $\mathbb{C}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{bmatrix}^T, a_2 = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{bmatrix}^T\}$ and $B = \{b_1 = \begin{bmatrix}\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}}\end{bmatrix}^T, b_2 = \begin{bmatrix}\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0\end{bmatrix}^T, b_3 = \begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{bmatrix}^T\}$. Verify that A and B form orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let $A^* = \{a_1^*, a_2^*\}$ and $B^* = \{b_1^*, b_2^*, b_3^*\}$. Recall that A^* and B^* are dual bases of V^* and W^* respectively. $\forall\ i \in \{1,2\}\ \forall\ j \in \{1,2,3\}\ \forall\ x \in \mathbb{C}^2\ \forall\ y \in \mathbb{C}^3$,

$$\begin{aligned}\left[a_i^* \otimes b_j^*\right](x, y) &= \left(a_i, x\right)\left(b_j, y\right) \\ &= \left(\overline{{}^{A}a_i} \odot {}^{A}x\right) \cdot \left(\overline{{}^{B}b_j} \odot {}^{B}y\right) \\ &= {}^{A}x\left[i\right] \cdot {}^{B}y\left[j\right]\end{aligned}$$

Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq 2, 1 \leq j \leq 3\}$. Above theorem implies that $A \otimes B$ is a basis of $\mathcal{L}\left(\mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}\right)$. Consider the following bi-linear function,

$$u\left(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T\right) = (x_1 + 2i \cdot x_2) \cdot (3i \cdot y_1 + y_3)$$

where $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{C}^2$ and $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T \in \mathbb{C}^3$.

It is straight forward to verify that u is bi-linear. Since A and B form bases of V and W respectively we get that,

$\forall \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{C}^2$ there exist unique α_1, $\alpha_2 \in \mathbb{C}$ such that,

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \alpha_1 a_1 + \alpha_2 a_2$$

$\forall \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T \in \mathbb{C}^3$ there exist unique β_1, β_2, $\beta_3 \in \mathbb{C}$ such that,

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^* = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3$$

Since u is bi-linear we get that,

$$u\left(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T\right) = u\left(\sum_{i=1}^{2} \alpha_i a_i, \sum_{j=1}^{3} \beta_j b_j\right) = \sum_{i=1}^{2}\sum_{j=1}^{3} \alpha_i \beta_j u\left(a_i, b_j\right)$$

It is quite clear that computing $u\left(a_i, b_j\right)$ is sufficient to determine the action of u on any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{C}^2$ $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T \in \mathbb{C}^3$.

$$\implies u\left(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T\right) = \sum_{i=1}^{2}\sum_{j=1}^{3} u\left(a_i, b_j\right)\left[a_i^* \otimes b_j^*\right](x, y)$$

With this illustration observe how $A \otimes B$ works as a basis of $\mathcal{L}\left(\mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}\right)$ and also note that the values of $u\left(a_i, b_j\right)$ is sufficient to compute $u\left(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T\right)$ for any $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{C}^2$ and $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T \in \mathbb{C}^3$.

2.2.4 Basis Transformation

Definition 2.2.3. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V) = n$ and $dim\,(W) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V and $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of W. Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Theorem 2.2.4 implies that $A \otimes B$ forms a basis of $\mathcal{L}\,(V \times W \to \mathbb{C})$. Hence $\forall\ u \in \mathcal{L}\,(V \times W \to \mathbb{C})$,

$$u = \sum_{i=1}^{n} \sum_{j=1}^{m} u\,(a_i, b_j)\,\left[a_i^* \otimes b_j^*\right]$$

Define coordinates of the bi-linear function u as follows,

$${}_{A\otimes B}u = \begin{bmatrix} u\,(a_1, b_1) & u\,(a_1, b_2) & . & . & . & u\,(a_1, b_m) \\ u\,(a_2, b_1) & u\,(a_2, b_2) & . & . & . & u\,(a_2, b_m) \\ . & . & . & . & . & . \\ u\,(a_n, b_1) & u\,(a_n, b_2) & . & . & . & u\,(a_n, b_m) \end{bmatrix}$$

Note :

1. Note that column vector representation is used for $1-$tensors and matrix representation is used for $2-$tensors.

Theorem 2.2.6. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V) = n$ and $dim\,(W) = m$. Let $A = \{a_1, ..., a_n\}$ and $C = \{c_1, ..., c_n\}$ be any two orthonormal basis of V. Let $B = \{b_1, ..., b_m\}$ and $D = \{d_1, ..., d_m\}$ be any two orthonormal basis of W. Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and $C \otimes D = \{c_i^* \otimes d_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Theorem 2.2.4 implies that $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}\,(V \times W \to \mathbb{C})$. Let $M \in \mathbb{C}^{n \times n}$ be the transformation matrix from basis A to C i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix} M$$

Let $N \in \mathbb{C}^{m \times m}$ be the transformation matrix from basis B to D i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & . & . & . & d_m \end{bmatrix} N$$

$\forall\ u \in \mathcal{L}(V \times W \to \mathbb{C})$,

$$\boxed{{}^{C\otimes D}u = \overline{M} \cdot {}^{A\otimes B}u \cdot N^*}$$

Proof. $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{C})$ since $A \otimes B$, $C \otimes D$ form bases of $\mathcal{L}(V \times W \to \mathbb{C})$ we get that,

$$u = \sum_{i=1}^{n}\sum_{j=1}^{m} u(a_i, b_j)\left[a_i^* \otimes b_j^*\right] = \sum_{p=1}^{n}\sum_{q=1}^{m} u(c_p, d_q)\left[c_p^* \otimes d_q^*\right]$$

$$\implies {}^{A\otimes B}u = \begin{bmatrix} u(a_1, b_1) & . & . & u(a_1, b_m) \\ . & . & . & . \\ u(a_n, b_1) & . & . & u(a_n, b_m) \end{bmatrix} \qquad {}^{C\otimes D}u = \begin{bmatrix} u(c_1, d_1) & . & . & u(c_1, d_m) \\ . & . & . & . \\ u(c_n, d_1) & . & . & u(c_n, d_m) \end{bmatrix}$$

Since M is the transformation matrix from basis A to C, we get that M^* is the transformation matrix from basis C to A i.e,

$$\begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} M^*$$

$\forall\ p \in \{1, 2, ..., n\}$,

$$c_p = \sum_{i=1}^{n} M_{ip}^* a_i = \sum_{i=1}^{n} \overline{M}_{pi} a_i$$

Since N is the transformation matrix from basis B to D, we get that N^* is the transformation matrix from basis D to B i.e,

$$\begin{bmatrix} d_1 & d_2 & . & . & . & d_m \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} N^*$$

$\forall\ q \in \{1, 2, ..., m\}$,

$$d_q = \sum_{j=1}^{m} N_{jq}^* b_j = \sum_{j=1}^{m} \overline{N}_{qj} \cdot b_j$$

Since u is bi-linear we get that,

$$u(c_p, d_q) = u\left(\sum_{i=1}^{n} \overline{M}_{pi} a_i, \sum_{j=1}^{m} \overline{N}_{qj} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{M}_{pi}\overline{N}_{qj} u(a_i, b_j)$$

$$\implies u\left(c_p, d_q\right) = \begin{bmatrix} \overline{M}_{p1} & . & . & \overline{M}_{pn} \end{bmatrix} \begin{bmatrix} u\left(a_1, b_1\right) & . & . & u\left(a_1, b_m\right) \\ . & . & . & . \\ u\left(a_n, b_1\right) & . & . & u\left(a_n, b_m\right) \end{bmatrix} \begin{bmatrix} \overline{N}_{q1} \\ . \\ . \\ \overline{N}_{qm} \end{bmatrix}$$

$$\begin{bmatrix} u\left(c_1, d_1\right) & .. & u\left(c_1, d_m\right) \\ . & .. & . \\ u\left(c_n, d_1\right) & .. & u\left(c_n, d_m\right) \end{bmatrix} = \begin{bmatrix} \overline{M}_{11} & .. & \overline{M}_{1n} \\ . & .. & . \\ \overline{M}_{n1} & .. & \overline{M}_{nn} \end{bmatrix} \begin{bmatrix} u\left(a_1, b_1\right) & .. & u\left(a_1, b_m\right) \\ . & .. & . \\ u\left(a_n, b_1\right) & .. & u\left(a_n, b_m\right) \end{bmatrix} \begin{bmatrix} \overline{N}_{11} & .. & \overline{N}_{m1} \\ . & .. & . \\ \overline{N}_{1m} & .. & \overline{N}_{mm} \end{bmatrix}$$

$$\implies {}^{C\otimes D}u = \overline{M} \cdot {}^{A\otimes B}u \cdot N^*$$

□

Illustration :
Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$ and $B = \{b_1 = \begin{bmatrix} \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \end{bmatrix}^T, b_2 = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{bmatrix}^T, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T\}$. Verify that A and B form orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let $A^* = \{a_1^*, a_2^*\}$ and $B^* = \{b_1^*, b_2^*, b_3^*\}$. Recall that A^* and B^* are dual bases of V^* and W^* respectively. $\forall\ i \in \{1,2\}\ \forall\ j \in \{1,2,3\}\ \forall\ x \in \mathbb{C}^2\ \forall\ y \in \mathbb{C}^3$,

$$\begin{aligned} \left[a_i^* \otimes b_j^*\right](x, y) &= (a_i, x)(b_j, y) = \left(\overline{{}^{A}a_i} \odot {}^{A}x\right) \cdot \left(\overline{{}^{B}b_j} \odot {}^{B}y\right) \\ &= {}^{A}x\,[i] \cdot {}^{B}y\,[j] \end{aligned}$$

Let $C = \{c_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, c_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$ and $D = \{d_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, d_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, d_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T\}$. Verify that C and D form orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let $C^* = \{c_1^*, c_2^*\}$ and $D^* = \{d_1^*, d_2^*, d_3^*\}$. Recall that C^* and D^* are dual bases of V^* and W^* respectively. $\forall\ p \in \{1,2\}\ \forall\ q \in \{1,2,3\}\ \forall\ x \in \mathbb{C}^2\ \forall\ y \in \mathbb{C}^3$,

$$\begin{aligned} \left[c_p^* \otimes d_q^*\right](x, y) &= (c_p, x)(d_q, y) = \left(\overline{{}^{C}c_p} \odot {}^{C}x\right) \cdot \left(\overline{{}^{D}d_q} \odot {}^{D}y\right) \\ &= {}^{C}x\,[i] \cdot {}^{D}y\,[j] \end{aligned}$$

Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq 2, 1 \leq j \leq 3\}$ and $C \otimes D = \{c_p^* \otimes d_q^* \mid 1 \leq$

$p \leq 2, 1 \leq q \leq 3\}$. Above theorem implies that $A \otimes B$ and $C \otimes D$ is a basis of $\mathcal{L}\left(\mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}\right)$.

Computing the basis transformation matrix from basis A to C

$$a_1 = \frac{1}{\sqrt{2}}c_1 + \frac{i}{\sqrt{2}}c_2 \qquad a_2 = \frac{1}{\sqrt{2}}c_1 - \frac{i}{\sqrt{2}}c_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to C of $\mathbb{C}^2$ which implies that M^* is the transformation matrix from basis C to A i.e,

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$\implies c_1 = \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 \qquad c_2 = -\frac{i}{\sqrt{2}}a_1 + \frac{i}{\sqrt{2}}a_2$$

Computing the basis transformation matrix from basis B to D

$$b_1 = \frac{i}{\sqrt{3}}d_1 + \frac{i}{\sqrt{3}}d_2 + \frac{i}{\sqrt{3}}d_3 \qquad b_2 = \frac{i}{\sqrt{2}}d_1 - \frac{i}{\sqrt{2}}d_2 \qquad b_1 = \frac{1}{\sqrt{6}}d_1 + \frac{1}{\sqrt{6}}d_2 - \frac{2}{\sqrt{6}}d_3$$

$$\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \implies N = \begin{bmatrix} \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

We get that N is the transformation matrix from basis B to D of $\mathbb{C}^3$ which implies that N^* is the transformation matrix from basis D to B i.e,

$$\begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$\implies d_1 = -\frac{i}{\sqrt{3}}b_1 - \frac{i}{\sqrt{2}}b_2 + \frac{1}{\sqrt{6}}b_3 \qquad d_2 = -\frac{i}{\sqrt{3}}b_1 + \frac{i}{\sqrt{2}}b_2 + \frac{1}{\sqrt{6}}b_3 \qquad d_3 = -\frac{i}{\sqrt{3}}b_1 - \frac{2}{\sqrt{6}}b_3$$

$\forall\, u \in \mathcal{L}\left(\mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}\right)$ since u is bi-linear $\forall\, p \in \{1,2\}$ $\forall\, q \in \{1,2,3\}$,

$$\begin{aligned} u\left(c_p, d_q\right) &= u\left(\sum_{i=1}^{2} M^*_{ip} a_i, \sum_{j=1}^{3} N^*_{jq} b_i\right) \\ &= \sum_{i=1}^{2}\sum_{j=1}^{3} \overline{M}_{pi}\overline{N}_{qj} u\left(a_i, b_j\right) \\ &= \begin{bmatrix} \overline{M}_{p1} & \overline{M}_{p2} \end{bmatrix} \begin{bmatrix} u\left(a_1, b_1\right) & u\left(a_1, b_2\right) & u\left(a_1, b_3\right) \\ u\left(a_2, b_1\right) & u\left(a_2, b_2\right) & u\left(a_2, b_3\right) \end{bmatrix} \begin{bmatrix} \overline{N}_{q1} \\ \overline{N}_{q2} \\ \overline{N}_{q3} \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} u\left(c_1, d_1\right) & u\left(c_1, d_2\right) & u\left(c_1, d_3\right) \\ u\left(c_2, d_1\right) & u\left(c_2, d_2\right) & u\left(c_2, d_3\right) \end{bmatrix} = \begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} \\ \overline{M}_{21} & \overline{M}_{22} \end{bmatrix} \begin{bmatrix} u\left(a_1, b_1\right) & u\left(a_1, b_2\right) & u\left(a_1, b_3\right) \\ u\left(a_2, b_1\right) & u\left(a_2, b_2\right) & u\left(a_2, b_3\right) \end{bmatrix} \begin{bmatrix} \overline{N}_{11} & \overline{N}_{21} & \overline{N}_{31} \\ \overline{N}_{12} & \overline{N}_{22} & \overline{N}_{32} \\ \overline{N}_{13} & \overline{N}_{23} & \overline{N}_{33} \end{bmatrix}$$

$$\implies {}^{C\otimes D}u = \overline{M} \cdot {}^{A\otimes B}u \cdot N^*$$

2.2.5 Invariance of computation of $2-$tensor under any orthonormal basis transformations

Theorem 2.2.7. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\left(V\right) = n$ and $dim\left(W\right) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V. Let $B = \{b_1, b_2, ..., b_m\}$ be any orthonormal basis of W. Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \le i \le n, 1 \le j \le m\}$. Theorem 2.2.4 implies that $A \otimes B$ forms basis of $\mathcal{L}\left(V \times W \to \mathbb{C}\right)$. $\forall\, u \in \mathcal{L}\left(V \times W \to \mathbb{C}\right)$ $\forall\, x \in V$ $\forall\, y \in W$,

$$\boxed{u\left(x, y\right) = \sum_{r=1}^{n}\sum_{s=1}^{m} {}^{A}x\left[r\right] \cdot {}^{A\otimes B}u\left[r, s\right] \cdot {}^{B}y\left[s\right] = \left({}^{A}x\right)^T \cdot {}^{A\otimes B}u \cdot {}^{B}y}$$

Note that ${}^{A\otimes B}u\left[r, s\right]$ is used to denote $u\left(a_r, b_s\right)$.

Proof. $\forall\, x \in V$ since A is a basis of V there exist unique $\alpha_r \in \mathbb{C}$ such that,

$$x = \sum_{r=1}^{n} \alpha_r a_r$$

$\forall\ y \in W$ since B is a basis of W there exist unique $\beta_s \in \mathbb{C}$ such that,

$$y = \sum_{s=1}^{m} \beta_s b_s$$

$\forall\ u \in \mathcal{L}(V \times W \to \mathbb{C})$ we get that,

$$u(x,y) = u\left(\sum_{r=1}^{n} \alpha_r a_r, \sum_{s=1}^{m} \beta_s b_s\right) = \sum_{r=1}^{n}\sum_{s=1}^{m} \alpha_r \beta_s u(a_r, b_s)$$

Since A and B form bases of V and W respectively we get that,

$$\alpha_r = {}^{A}x[r] \qquad\qquad \beta_s = {}^{B}y[s]$$

Since $A \otimes B$ forms basis of $\mathcal{L}(V \times W \to \mathbb{C})$ we get that,

$${}^{A\otimes B}u = \begin{bmatrix} u(a_1,b_1) & u(a_1,b_2) & . & . & . & u(a_1,b_m) \\ u(a_2,b_1) & u(a_2,b_2) & . & . & . & u(a_2,b_m) \\ . & . & . & . & . & . \\ u(a_n,b_1) & u(a_n,b_2) & . & . & . & u(a_n,b_m) \end{bmatrix}$$

$$\implies u(x,y) = \sum_{r=1}^{n}\sum_{s=1}^{m} {}^{A}x[r] \cdot {}^{A\otimes B}u[r,s] \cdot {}^{B}y[s]$$

□

Remark :

1. Let $A = \{a_1, a_2, ..., a_n\}$ and $C = \{c_1, c_2, ..., c_n\}$ be any two orthonormal basis of V. Let $B = \{b_1, b_2, ..., b_m\}$ and $D = \{d_1, d_2, ..., d_m\}$ be any two orthonormal basis of W. Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and $C \otimes D = \{c_i^* \otimes d_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Theorem 2.2.4 implies that $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}(V \times W \to \mathbb{C})$. $\forall\ u \in \mathcal{L}(V \times W \to \mathbb{C})$ $\forall$ $x \in V$ $\forall\ y \in W$ we get that,

$$u(x,y) = \left({}^{A}x\right)^T \cdot {}^{A\otimes B}u \cdot {}^{B}y = \left({}^{C}x\right)^T \cdot {}^{C\otimes D}u \cdot {}^{D}y$$

2. It is easy to observe that $\forall\ (x,y) \in V \times W$, $u(x,y)$ can be determined by the action of ${}^{A\otimes B}u$ on ${}^{A}x$ and ${}^{B}y$. Hence if we fix computations with respect to orthonormal bases A and B of V and W respectively we can identify u with ${}^{A\otimes B}u$.

3. Let $dim\,(V) = n$, $dim\,(W) = m$. $\forall\ u \in \mathcal{L}\,(V \times W \to \mathbb{C})$,

$$
{}^{A\otimes B}u = \begin{bmatrix} u(a_1,b_1) & u(a_1,b_2) & . & . & . & u(a_1,b_m) \\ u(a_2,b_1) & u(a_2,b_2) & . & . & . & u(a_2,b_m) \\ . & . & . & . & . & . \\ u(a_n,b_1) & u(a_n,b_2) & . & . & . & u(a_n,b_m) \end{bmatrix}
$$

   Hence, $2-$fold tensor product space $\mathcal{L}\,(V \times W \to \mathbb{C})$ is isomorphic to $\mathbb{C}^{n\times m}$. (It is straight-forward to verify and is left to reader. For the proof technique you may refer lemma 1.1.9)

4.

$$
\begin{aligned}
u(x,y) &= \left({}^{C}x\right)^T \cdot {}^{C\otimes D}u \cdot {}^{D}y = \left(M \cdot {}^{A}x\right)^T \cdot \overline{M} \cdot {}^{A\otimes B}u \cdot N^* \cdot \left(N \cdot {}^{B}y\right) \\
&= \left({}^{A}x\right)^T \cdot M^T \cdot \overline{M} \cdot {}^{A\otimes B}u \cdot N^* \cdot N \cdot {}^{B}y
\end{aligned}
$$

   Since $M^*M = N^*N = I$ we get,

$$
\Longrightarrow \boxed{u(x,y) = \left({}^{A}x\right)^T \cdot {}^{A\otimes B}u \cdot {}^{B}y = \left({}^{C}x\right)^T \cdot {}^{C\otimes D}u \cdot {}^{D}y}
$$

Illustration :
Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$ and $B = \{b_1 = \begin{bmatrix} \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \end{bmatrix}^T, b_2 = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{bmatrix}^T, b_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T\}$. Verify that A and B form orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let $A^* = \{a_1^*, a_2^*\}$ and $B^* = \{b_1^*, b_2^*, b_3^*\}$. Recall that A^* and B^* are dual bases of V^* and W^* respectively. $\forall\ i \in \{1,2\}$ $\forall\ j \in \{1,2,3\}$ $\forall\ x \in \mathbb{C}^2$ $\forall\ y \in \mathbb{C}^3$,

$$
\begin{aligned}
\left[a_i^* \otimes b_j^*\right](x,y) &= (a_i, x)\,(b_j, y) = \left(\overline{{}^{A}a_i} \odot {}^{A}x\right) \cdot \left(\overline{{}^{B}b_j} \odot {}^{B}y\right) \\
&= {}^{A}x\,[i] \cdot {}^{B}y\,[j]
\end{aligned}
$$

Let $C = \{c_1 = \begin{bmatrix}1 & 0\end{bmatrix}^T, a_2 = \begin{bmatrix}0 & 1\end{bmatrix}^T\}$ and $D = \{d_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, b_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T, b_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T\}$. Verify that C and D form orthonormal basis of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Let $C^* = \{c_1^*, c_2^*\}$ and $D^* = \{d_1^*, d_2^*, d_3^*\}$. Recall that C^* and D^* are dual bases of V^* and W^* respectively. $\forall\ p \in \{1,2\}\ \forall\ q \in \{1,2,3\}\ \forall\ x \in \mathbb{C}^2\ \forall\ y \in \mathbb{C}^3$,

$$\begin{aligned}\left[c_p^* \otimes d_q^*\right](x,y) &= (c_p, x)(d_q, y) = \left(\overline{{}^C c_p} \odot {}^C x\right) \cdot \left(\overline{{}^D d_q} \odot {}^D y\right)\\ &= {}^C x\,[i] \cdot {}^D y\,[j]\end{aligned}$$

Let $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \le i \le 2, 1 \le j \le 3\}$ and $C \otimes D = \{c_p^* \otimes d_q^* \mid 1 \le p \le 2, 1 \le q \le 3\}$. Above theorem implies that $A \otimes B$ and $C \otimes D$ is a basis of $\mathcal{L}\left(\mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C}\right)$. In the previous illustration we have already shown that the transformation matrix M from basis A to C is

$$M = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{bmatrix}$$

We have also seen that $\forall\ x \in \mathbb{C}^2$,

$${}^C x = M \cdot {}^A x$$

$$N = \begin{bmatrix}\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{i}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{bmatrix}$$

We have also seen that $\forall\ y \in \mathbb{C}^3$,

$${}^D y = N \cdot {}^B y$$

Since C and D are bases of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively we get that,

$$x = {}^C x\,[1]\,c_1 + {}^C x\,[2]\,c_2 \qquad y = {}^D y\,[1]\,d_1 + {}^D y\,[2]\,d_2 + {}^D y\,[3]\,d_3$$

Since u is bi-linear we get,

$$u(x,y) = u\left(\sum_{i=1}^{2} {}^{C}x[i]\, c_i, \sum_{j=1}^{3} {}^{D}y[j]\, d_j\right) = \sum_{i=1}^{2}\sum_{j=1}^{3} {}^{C}x[i]\, {}^{D}y[j]\, u(c_i, d_j)$$

$$\implies u(x,y) = \begin{bmatrix} {}^{C}x[1] & {}^{C}x[2] \end{bmatrix} \begin{bmatrix} u(c_1,d_1) & u(c_1,d_2) & u(c_1,d_3) \\ u(c_2,d_1) & u(c_2,d_2) & u(c_2,d_3) \end{bmatrix} \begin{bmatrix} {}^{B}y[1] \\ {}^{B}y[2] \\ {}^{B}y[3] \end{bmatrix}$$

$\forall\ u \in \mathcal{L}(\mathbb{C}^2 \times \mathbb{C}^3 \to \mathbb{C})$ we have already shown in previous illustration that,

$$^{C\otimes D}u = \overline{M} \cdot {}^{A\otimes B}u \cdot N^*$$

$$\begin{aligned} u(x,y) &= \left({}^{C}x\right)^T \cdot {}^{C\otimes D}u \cdot {}^{D}y = \left(M \cdot {}^{A}x\right)^T \cdot \overline{M} \cdot {}^{A\otimes B}u \cdot N^* \cdot \left(N \cdot {}^{B}y\right) \\ &= \left({}^{A}x\right)^T \cdot M^T \cdot \overline{M} \cdot {}^{A\otimes B}u \cdot N^* \cdot N \cdot {}^{B}y \end{aligned}$$

Since M and N are orthogonal matrices we get that,

$$u(x,y) = \left({}^{A}x\right)^T \cdot {}^{A\otimes B}u \cdot {}^{B}y = \left({}^{C}x\right)^T \cdot {}^{C\otimes D}u \cdot {}^{D}y$$

2.2.6 Inner products on 2-fold tensor product spaces

In this section we define a function $() : V \otimes W \times V \otimes W \to \mathbb{C}$ in terms of inner products defined on V^* and W^* and prove that this function is an inner product.

Definition 2.2.4. Let V, W be any finite dimensional inner product space where $\dim(V) = n$ and $\dim(W) = m$. $\forall\ u_1, ..., u_k, w_1, ..., w_l \in V^*\ \forall\ v_1, ..., v_k, t_1, ..., t_l \in W^*\ \forall\ \alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l \in \mathbb{C}$ Define the following function $() : V \otimes W \times V \otimes W \to \mathbb{C}$,

$$\boxed{\left(\sum_{i=1}^{k} \alpha_i [u_i \otimes v_i], \sum_{j=1}^{l} \beta_j [w_j \otimes t_j]\right) = \sum_{i=1}^{k}\sum_{j=1}^{l} \overline{\alpha}_i \beta_j (u_i, w_j)_1 (v_i, t_j)_2} \qquad (2.1)$$

Note that $()_1$ is an inner product on V^* and $()_2$ is an inner product on W^*. In the subsequent analysis we drop the subscripts since we believe that the context of usage shall be clear.

In this lemma it is shown that how above definition can be used to compute $(u, v)\ \forall\ u, v \in V \otimes W$.

Lemma 2.2.8. Let V, W be any finite dimensional inner product space where $\dim(V) = n$ and $\dim(W) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V and $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ be the corresponding dual basis. Let $B = \{b_1, b_2, ..., b_m\}$ be any orthonormal basis of W and $B^* = \{b_1^*, b_2^*, ..., b_m^*\}$ be the corresponding dual basis. $\forall\ u, v \in V \otimes W$ since $A \otimes B$ is a basis of $V \otimes W$ there exist $\alpha_{ij}, \beta_{pq} \in \mathbb{C}$ such that,

$$u = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \left[a_i^* \otimes b_j^*\right] \qquad v = \sum_{p=1}^{n} \sum_{q=1}^{m} \beta_{pq} \left[a_p^* \otimes b_q^*\right]$$

$$\boxed{(u, v) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{q=1}^{m} \overline{\alpha}_{ij} \beta_{pq} \left(a_i^*, a_p^*\right) \left(b_j^*, b_q^*\right)} \tag{2.2}$$

Proof. Proof is straight forward and left to reader (use equation 3.1.1). □

Remark :

1. The existence of inner products on dual space i.e, $()_1$ and $()_2$ is already shown in the previous chapter section 2.1.6. Also It is already shown that with respect to the inner product defined in section 2.1.6 A^* and B^* are orthonormal bases which implies that,

$$(u, v) = \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{\alpha}_{ij} \beta_{ij} \tag{2.3}$$

2. Note that in the subsequent analysis we consider arbitrary inner products on V^* and W^* in order to make the theory more general. Hence equation 3.1.1 is used instead of equation 3.1.1.

Lemma 2.2.9. $\forall\ u, v \in V \otimes W$,

$$(u, v) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{q=1}^{m} \overline{\alpha}_{ij} \beta_{pq} \left(a_i^*, a_p^*\right) \left(b_j^*, b_q^*\right) \quad \text{is well-defined}$$

Proof. Let $C = \{c_1, c_2, ..., c_n\}$ be any another orthonormal basis of V and $C^* = \{c_1^*, c_2^*, ..., c_n^*\}$ be the corresponding dual basis. Let $D = \{d_1, d_2, ..., d_m\}$ be any another orthonormal basis of W and $D^* = \{d_1^*, d_2^*, ..., d_m^*\}$ be the corresponding dual basis. Since $C \otimes D$ is a basis of $V \otimes W$ there exist $\gamma_{kl}, \delta_{rs} \in \mathbb{C}$ such that,

$$u = \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{kl} \left[c_k^* \otimes d_l^*\right] \qquad\qquad v = \sum_{r=1}^{n} \sum_{s=1}^{m} \delta_{rs} \left[c_r^* \otimes d_s^*\right]$$

Inner product of u and v using basis C^* and D^* is

$$(u, v) = \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{m} \overline{\gamma}_{kl} \delta_{rs} \left(c_k^*, c_r^*\right) \left(d_l^*, d_s^*\right)$$

To claim (u, v) is well-defined it is enough to show that

$$\sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{m} \overline{\gamma}_{kl} \delta_{rs} \left(c_k^*, c_r^*\right) \left(d_l^*, d_s^*\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{q=1}^{m} \overline{\alpha}_{ij} \beta_{pq} \left(a_i^*, a_p^*\right) \left(b_j^*, b_q^*\right)$$

Let $M \in \mathbb{C}^{n \times n}$ be the transformation matrix from basis A to C. Theorem 2.1.5 implies that,

$$\begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} = \begin{bmatrix} c_1^* & c_2^* & . & . & c_n^* \end{bmatrix} \overline{M} \implies a_i^* = \sum_{k=1}^{n} \overline{M}_{ki} c_k^* \text{ and } a_p^* = \sum_{r=1}^{n} \overline{M}_{rp} c_r^*$$

Let $N \in \mathbb{C}^{m \times m}$ be the transformation matrix from basis B to D. Theorem 2.1.5 implies that,

$$\begin{bmatrix} b_1^* & b_2^* & . & . & b_m^* \end{bmatrix} = \begin{bmatrix} d_1^* & d_2^* & . & . & d_m^* \end{bmatrix} \overline{N} \implies b_j^* = \sum_{l=1}^{m} \overline{N}_{lj} d_l^* \text{ and } b_q^* = \sum_{s=1}^{m} \overline{N}_{sq} d_s^*$$

Above two equations imply that,

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{q=1}^{m} \overline{\alpha}_{ij} \beta_{pq} \left(a_i^*, a_p^*\right) \left(b_j^*, b_q^*\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{q=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{m} \overline{\alpha}_{ij} \beta_{pq} M_{ki} \overline{M}_{rp} N_{lj} \overline{N}_{sq} \left(c_k^*, c_r^*\right) \left(d_l^*, d_s^*\right) \end{aligned}$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{m}\sum_{r=1}^{n}\sum_{s=1}^{m}\left(\sum_{i=1}^{n}\sum_{j=1}^{m} M_{ki}\overline{\alpha}_{ij}N_{lj}\right)\left(\sum_{p=1}^{n}\sum_{q=1}^{m}\overline{M}_{rp}\beta_{pq}\overline{N}_{sq}\right)(c_k^*, c_r^*)(d_l^*, d_s^*)$$

Note that

$$^{C\otimes D}u[k,l] = \gamma_{kl} \qquad ^{C\otimes D}v[r,s] = \delta_{rs} \qquad ^{A\otimes B}u[i,j] = \alpha_{ij} \qquad ^{A\otimes B}v[p,q] = \beta_{pq}$$

From theorem 3.2.4 we get,

$$^{C\otimes D}u = \overline{M}\cdot {}^{A\otimes B}u \cdot N^* \qquad\qquad ^{C\otimes D}v = \overline{M}\cdot {}^{A\otimes B}v \cdot N^*$$

$$\implies \gamma_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{M}_{ki}\alpha_{ij}N_{jl}^* = \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{M}_{ki}\alpha_{ij}\overline{N}_{lj} \text{ and}$$

$$\delta_{rs} = \sum_{p=1}^{n}\sum_{q=1}^{m}\overline{M}_{rp}\beta_{pq}N_{qs}^* = \sum_{p=1}^{n}\sum_{q=1}^{m}\overline{M}_{rp}\beta_{pq}\overline{N}_{sq}$$

$$\implies \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{m}\overline{\alpha}_{ij}\beta_{pq}\left(a_i^*, a_p^*\right)\left(b_j^*, b_q^*\right) = \sum_{k=1}^{n}\sum_{l=1}^{m}\sum_{r=1}^{n}\sum_{s=1}^{m}\overline{\gamma}_{kl}\delta_{rs}(c_k^*, c_r^*)(d_l^*, d_s^*)$$

□

Lemma 2.2.10. $\forall\, u, v \in V \otimes W$,

$$(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{m}\overline{\alpha}_{ij}\beta_{pq}\left(a_i^*, a_p^*\right)\left(b_j^*, b_q^*\right) \quad \text{is an inner product}$$

Proof. **Linearity :** $\forall\, u, v, w \in V \otimes W\ \forall\, \delta \in \mathbb{C}$,

$$\begin{aligned}(u, v+w) &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}a_i^*\otimes b_j^*, \sum_{p=1}^{n}\sum_{q=1}^{m}(\beta_{pq}+\gamma_{pq})\,a_p^*\otimes b_q^*\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{m}\overline{\alpha}_{ij}(\beta_{pq}+\gamma_{pq})\left(a_i^*, a_p^*\right)\left(b_j^*, b_q^*\right) = (u,v)+(u,w)\end{aligned}$$

$$\begin{aligned}(u, \delta v) &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}a_i^*\otimes b_j^*, \sum_{p=1}^{n}\sum_{q=1}^{m}(\delta\beta_{pq})\,a_p^*\otimes b_q^*\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{m}\overline{\alpha}_{ij}\delta\beta_{pq}\left(a_i^*, a_p^*\right)\left(b_j^*, b_q^*\right) = \delta\,(u,v)\end{aligned}$$

Conjugate Symmetry : $\forall\ u, v \in V \otimes W$,

$$(u, v) = \overline{\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{m} \overline{\alpha}_{ij}\beta_{pq}\left(a_i^*, a_p^*\right)\left(b_j^*, b_q^*\right)}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{m} \overline{\beta}_{pq}\alpha_{ij}\beta_{pq}\left(a_p^*, a_i^*\right)\left(b_q^*, b_j^*\right) = \overline{(v, u)}$$

Positive Definiteness : Since both V^*, W^* over $\mathbb{C}$ are inner product spaces using Gram Schmidt process there exist orthonormal basis for both V^* and W^*. Without loss of generality assume A^* and B^* form orthonormal bases of V^* and W^* respectively.

$$\begin{aligned}(u, u) = 0 \iff & (\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\left[a_i^* \otimes a_j^*\right], \sum_{p=1}^{n}\sum_{q=1}^{m} \alpha_{pq}\left[a_p^* \otimes a_q^*\right]) = 0 \\ \iff & \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{\alpha}_{ij}\alpha_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} \left|\alpha_{ij}\right|^2 = 0 \\ \iff & u = 0\end{aligned}$$

□

We end this section showing how inner products can be used in giving an alternate proof for the linear independence of the set $A \otimes B$. Consider the inner product on dual space defined as in section 2.1.6. $\forall\ i, j \in \{1, 2, ..., n\}\ \forall\ p, q \in \{1, 2, ..., m\}$,

$$\begin{aligned}\left(a_i^* \otimes b_p^*, a_j^* \otimes b_q^*\right) = \left(a_i^*, a_j^*\right)\left(b_p^*, b_q^*\right) = \left({}^{A^*}a_i^* \odot {}^{A^*}a_j^*\right)\left({}^{B^*}b_p^* \odot {}^{B^*}b_q^*\right) &= 1 \quad \text{if } (i, j) = (p, q) \\ &= 0 \quad \text{otherwise}\end{aligned}$$

From lemma 1.1.7 it is straight forward to verify linear independence of $A \otimes B$.

2.2.7 Linear operators on 2-fold tensor product spaces

Definition 2.2.5. Let $\mathcal{L}(V \otimes W)$ denote the set of all linear operators over the tensor product space of V and W $\mathcal{L}(V \times W \to \mathbb{C}) = V \otimes W$. Define addition and scalar multiplication on the set $\mathcal{L}(V \otimes W)$ as follows $\forall\ T, W \in \mathcal{L}(V \otimes W)\ \forall$

$u \in V \otimes W \ \forall\ \alpha \in \mathbb{C}$,

$$[T+W]\,u = T\,(u) + W\,(u)$$

$$[\alpha T]\,u = \alpha \cdot T\,(u)$$

Remark :

1. It is straight forward to verify that $\mathcal{L}\,(V \otimes W)$ is a vector space over $\mathbb{C}$ and is left to reader (refer lemma 2.1.1).

2. Note that $\mathcal{L}\,(V \otimes W)$ is also called tensor product space of operators on $V \otimes W$.

Next we define the tensor product of two operators and show that any operator on $V \otimes W$ can be expressed in terms of tensor products.

Definition 2.2.6. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $\dim(V) = n$ and $\dim(W) = m$. Let T be an operator on V^* and U be an operator on W^*. Define the tensor product of T and U as an operator on $V \otimes W$ i.e, $T \otimes U : V \otimes W \to V \otimes W \ \forall\ x_1, x_2, ..., x_k \in V^* \ \forall\ y_1, y_2, ..., y_k \in W^*$ $\forall\ \alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{C}$,

$$\boxed{[T \otimes U]\left(\sum_{i=1}^{k} \alpha_i x_i \otimes y_i\right) = \sum_{i=1}^{k} \alpha_i\,[T\,(x_i)] \otimes [W\,(y_i)]} \tag{2.4}$$

In this lemma it is shown that how above definition can be used to compute $[T \otimes U]\,(x) \ \forall\ x \in V \otimes W$.

Lemma 2.2.11. Let $A = \{a_1, a_2, ..., a_n\}$ be any orthonormal basis of V and $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ be the corresponding dual basis. Let $B = \{b_1, b_2, ..., b_m\}$ be any orthonormal basis of W and $B^* = \{b_1^*, b_2^*, ..., b_m^*\}$ be the corresponding dual basis. $\forall\ x \in V \otimes W$ since $A \otimes B$ is a basis of $V \otimes W$ there exist unique $\alpha_{ij} \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} a_i^* \otimes b_j^*$$

$\forall\, T \in \mathcal{L}(V^*)\ \forall\, U \in \mathcal{L}(W^*)$,

$$[T \otimes U](x) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\,[T(a_i^*)] \otimes \left[W\left(b_j^*\right)\right]$$

Proof. Proof is straight forward and left to reader (use equation 3.1.1). □

Remark :

1. $\forall\, T \in \mathcal{L}(V^*)\ \forall\, U \in \mathcal{L}(W^*)\ [T \otimes U]$ is bi-linear. $\forall\, x, y \in V \otimes W$ since $A \otimes B$ is a basis of $V \otimes W$ there exist unique $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} a_i^* \otimes b_j^* \qquad y = \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} a_i^* \otimes b_j^*$$

$\forall\, \gamma \in \mathbb{C}$,

$$\begin{aligned} [T \otimes U](x + \gamma y) &= [T \otimes U]\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_{ij} + \gamma\beta_{ij})\, a_i^* \otimes b_j^*\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_{ij} + \gamma\beta_{ij})\,[T(a_i^*)] \otimes \left[U\left(b_j^*\right)\right] \\ &= [T \otimes U](x) + \gamma\,[T \otimes U](y) \end{aligned}$$

2. $\forall\, T, U \in \mathcal{L}(V^*)\ \forall\, W \in \mathcal{L}(W^*)$,

$$[T + U] \otimes W = T \otimes W + U \otimes W$$

$\forall\, T \in \mathcal{L}(V^*)\ \forall\, U, W \in \mathcal{L}(W^*)$,

$$T \otimes [U + W] = T \otimes U + T \otimes W$$

$\forall\, T \in \mathcal{L}(V^*)\ \forall\, U \in \mathcal{L}(W^*)\ \forall\, \alpha \in \mathbb{C}$,

$$[\alpha T] \otimes U = T \otimes [\alpha U] = \alpha\,[T \otimes U]$$

These properties are straight forward to verify and are left to reader.

Lemma 2.2.12. $\forall\, T \in \mathcal{L}(V^*)\ \forall\, U \in \mathcal{L}(W^*)$, $[T \otimes U]$ is well-defined

Proof. Let $C = \{c_1, c_2, ..., c_n\}$ be any another orthonormal basis of V. Let $C^* = \{c_1^*, c_2^*, ..., c_n^*\}$ be the corresponding dual basis. Let $D = \{d_1, d_2, ..., d_m\}$ be any another orthonormal basis of W. Let $D^* = \{d_1^*, d_2^*, ..., d_m^*\}$ be the corresponding dual basis. Since $C \otimes D$ is a basis of $V \otimes W$ there exist unique $\beta_{pq} \in \mathbb{C}$ such that,

$$x = \sum_{p=1}^{n} \sum_{q=1}^{m} \beta_{pq} c_p^* \otimes d_q^*$$

Applying $x \in V \otimes W$ expressed in terms of basis C and D to the operator $[T \otimes U]$ we get that,

$$[T \otimes U]\,(x) = \sum_{p=1}^{n} \sum_{q=1}^{m} \beta_{pq} \left[T\left(c_p^*\right)\right] \otimes \left[U\left(d_q^*\right)\right]$$

To claim $(T \otimes U)$ is well-defined it is enough to show that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} \left[T\left(a_i^*\right)\right] \otimes \left[U\left(b_j^*\right)\right] = \sum_{p=1}^{n} \sum_{q=1}^{m} \beta_{pq} \left[T\left(c_p^*\right)\right] \otimes \left[U\left(d_q^*\right)\right]$$

Let $M \in \mathbb{C}^{n \times n}$ be the transformation matrix from basis A to C. Theorem 2.1.5 implies that,

$$\begin{bmatrix} a_1^* & a_2^* & . & . & a_n^* \end{bmatrix} = \begin{bmatrix} c_1^* & c_2^* & . & . & c_n^* \end{bmatrix} \overline{M} \implies a_i^* = \sum_{p=1}^{n} \overline{M}_{pi} c_p^* \implies T\left(a_i^*\right) = \sum_{p=1}^{n} \overline{M}_{pi} T\left(c_p^*\right)$$

Let $N \in \mathbb{C}^{m \times m}$ be the transformation matrix from basis B to D. Theorem 2.1.5 implies that,

$$\begin{bmatrix} b_1^* & b_2^* & . & . & b_m^* \end{bmatrix} = \begin{bmatrix} d_1^* & d_2^* & . & . & d_m^* \end{bmatrix} \overline{N} \implies b_j^* = \sum_{q=1}^{n} \overline{N}_{qj} d_q^* \text{ and } U\left(b_j^*\right) = \sum_{q=1}^{m} \overline{N}_{qj} U\left(d_q^*\right)$$

Above two equations imply that,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} \left[T\left(a_i^*\right)\right] \otimes \left[U\left(b_j^*\right)\right] = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{p=1}^{n} \sum_{q=1}^{m} \overline{M}_{pi} \alpha_{ij} \overline{N}_{qj} \left[T\left(c_p^*\right)\right] \otimes \left[U\left(d_q^*\right)\right]$$

$$= \sum_{p=1}^{n}\sum_{q=1}^{m}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\overline{M}_{pi}\alpha_{ij}\overline{N}_{qj}\right)\left[T\left(c_p^*\right)\right]\otimes\left[U\left(d_q^*\right)\right]$$

Note that

$$^{C\otimes D}u\,[p,q] = \beta_{pq} \qquad\qquad ^{A\otimes B}u\,[i,j] = \alpha_{ij}$$

From theorem 2.3.6 we get,

$$^{C\otimes D}u = \overline{M}\cdot{}^{A\otimes B}u\cdot N^* \qquad\qquad ^{C\otimes D}v = \overline{M}\cdot{}^{A\otimes B}v\cdot N^*$$

$$\implies \beta_{pq} = \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{M}_{pi}\alpha_{ij}N^*_{qj} = \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{M}_{pi}\alpha_{ij}\overline{N}_{jq}$$

$$\implies \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}\left[T\left(a_i^*\right)\right]\otimes\left[U\left(b_j^*\right)\right] = \sum_{p=1}^{n}\sum_{q=1}^{m}\beta_{pq}\left[T\left(c_p^*\right)\right]\otimes\left[U\left(d_q^*\right)\right]$$

□

Definition 2.2.7. Let V, W be any two finite dimensional inner product spaces over field $\mathbb{C}$ where $\dim(V) = n$ and $\dim(W) = m$. Let $A = \{a_1, a_2, ..., a_n\}$ be an orthonormal basis of V. Let $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ be the corresponding dual basis. Let $B = \{b_1, b_2, ..., b_m\}$ be an orthonormal basis of W. Let $B^* = \{b_1^*, b_2^*, ..., b_m^*\}$ be the corresponding dual basis. Let $T^* = \{T_{ij} \mid 1 \le i, j \le n\}$ where $\forall\ i, j, k \in \{1, 2, ..., n\}$ $T_{ij} \in \mathcal{L}(V^*)$ is defined as follows,

$$\begin{aligned} T_{ij}\left(a_k^*\right) &= a_j^* \qquad \text{if } k = i \\ &= 0 \qquad \text{if } k \neq i \end{aligned}$$

Let $U^* = \{U_{pq} \mid 1 \le p, q \le m\}$ where $\forall\ p, q, r \in \{1, 2, ..., m\}$ $U_{pq} \in \mathcal{L}(W^*)$ is defined as follows,

$$\begin{aligned} U_{pq}\left(b_r^*\right) &= b_r^* \qquad \text{if } p = r \\ &= 0 \qquad \text{if } p \neq r \end{aligned}$$

Define $T^* \otimes U^* = \{T_{ij} \otimes U_{pq} \mid 1 \le i, j \le n, 1 \le p, q \le m\}$. Note that each $T_{ij} \otimes U_{pq} \in T^* \otimes U^*$ is well-defined and a linear operator on $V \otimes W$.

Theorem 2.2.13. $T^* \otimes U^*$ forms a basis of $\mathcal{L}(V \otimes W)$.

Proof. **Span :**
$\forall\ x \in V \otimes W$ since $A \otimes B$ is a basis of $V \otimes W$ there exist unique $\alpha_{ip} \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n} \sum_{p=1}^{m} \alpha_{ip} a_i^* \otimes b_p^*$$

$\forall\ W \in \mathcal{L}(V \otimes W)$ since W is a linear operator we get,

$$W(x) = \sum_{i=1}^{n} \sum_{p=1}^{m} \alpha_{ip} W\left(a_i^* \otimes b_p^*\right)$$

Since W is an operator $\forall\ i \in \{1, 2, ..., n\}$ $\forall\ p \in \{1, 2, ..., m\}$ there exist $\beta_{jq} \in \mathbb{C}$ such that,

$$W\left(a_i^* \otimes b_p^*\right) = \sum_{j=1}^{n} \sum_{q=1}^{m} \beta_{i,p,j,q} a_j^* \otimes b_q^*$$

$$\implies W(x) = \sum_{i=1}^{n} \sum_{p=1}^{m} \sum_{j=1}^{n} \sum_{q=1}^{m} \beta_{i,p,j,q} \alpha_{ip} a_j^* \otimes b_q^*$$

Note that $\forall\ i, j \in \{1, 2, ..., n\}$ $\forall\ p, q \in \{1, 2, ..., m\}$ since $T_{ij} \otimes U_{pq}$ is linear,

$$\begin{aligned}
\left[T_{ij} \otimes U_{pq}\right](x) &= \left[T_{ij} \otimes U_{pq}\right]\left(\sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_{kl} a_k^* \otimes b_l^*\right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_{kl} \left[T_{ij}\left(a_k^*\right)\right] \otimes \left[U_{pq}\left(b_l^*\right)\right] \\
&= \alpha_{ip} a_j^* \otimes b_q^*
\end{aligned}$$

$$\implies W(x) = \sum_{p=1}^{n} \sum_{q=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{i,p,j,q} \left[T_{ij} \otimes U_{pq}\right](x)$$

$$\implies W = \sum_{p=1}^{n} \sum_{q=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{i,p,j,q} \left[T_{ij} \otimes U_{pq}\right]$$

$$\implies T^* \otimes U^* \text{ spans } \mathcal{L}(V \otimes W)$$

Linear Independence : $\forall\ i, j \in \{1, 2, ..., n\}\ \forall\ p, q \in \{1, 2, ..., m\}$. Consider,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{m}\sum_{q=1}^{m}\alpha_{i,p,j,q}T_{ij}\otimes U_{pq}=0$$

$\forall\ k \in \{1, 2, ..., n\}\ \forall\ l \in \{1, 2, ..., m\}$ applying $a_k^* \otimes b_l^*$ we get,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{m}\sum_{q=1}^{m}\alpha_{i,p,j,q}\left[T_{ij}\left(a_k^*\right)\right]\otimes\left[U_{pq}\left(b_l^*\right)\right]=0 \implies \sum_{j=1}^{n}\sum_{q=1}^{m}\alpha_{k,l,j,q}a_j^*\otimes b_q^*=0$$

Since $A \otimes B$ is a basis of $V \otimes W$ we get that,

$$\alpha_{k,l,j,q}=0 \qquad \forall\ j \in \{1,2,...,n\}\ \forall\ q \in \{1,2,...,m\}$$

$\implies T^* \otimes U^*$ is a linearly independent set and forms a basis of $\mathcal{L}(V \otimes W)$

□

Corollary 2.2.14.

$$\boxed{\dim\left(\mathcal{L}\left(V\otimes W\right)\right)=\left(\dim\left(V\otimes W\right)\right)^2}$$

2.3 k-fold tensor product spaces

2.3.1 Multi-linear Functions

Definition 2.3.1. Let V_1, V_2, ..., V_k be vector spaces over field $\mathbb{C}$ with inner products $()_1 : V_1 \times V_1 \to \mathbb{C}$, $()_2 : V_2 \times V_2 \to \mathbb{C}$, ..., $()_k : V_k \times V_k \to \mathbb{C}$ defined on V_1, V_2, ..., V_k respectively. Note that subscripts $()_1$, $()_2$, ..., $()_k$ will be dropped if the context is clear. A function $u : V_1 \times V_2 \times ... \times V_k \to \mathbb{C}$ is called multi-linear if the following holds,

1. $\forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_i, \tilde{x}_i \in V_i\ ...\ \forall\ x_k \in V_k$ where $1 \leq i \leq k$,

$$u(x_1, x_2, ..., x_i + \tilde{x}_i, ..., x_k) = u(x_1, x_2, ..., x_i, ..., x_k) + u(x_1, x_2, ..., \tilde{x}_i, ..., x_k)$$

2. $\forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_i \in V_i\ ...\ \forall\ x_k \in V_k$ where $1 \leq i \leq k\ \forall\ \alpha \in \mathbb{C}$,

$$u(x_1, x_2, ..., \alpha x_i, ..., x_k) = \alpha u(x_1, x_2, ..., x_k)$$

Let set $S = \{u : V_1 \times V_2 \times ... \times V_k \to \mathbb{C} \mid u \text{ is multi-linear}\}$. Define addition and multiplication on the set S as follows $\forall\ u, v \in S\ \forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_k \in V_k$ $\forall\ \alpha \in \mathbb{C}$,

$$[u + v](x_1, x_2, ..., x_k) = u(x_1, x_2, ..., x_k) + v(x_1, x_2, ..., x_k)$$

$$[\alpha u](x_1, x_2, ..., x_k) = \alpha u(x_1, x_2, ..., x_k)$$

Lemma 2.3.1. S is closed under addition and scalar multiplication.

Proof. **Claim 1 :** $\forall\ u, v \in S\ [u + v] \in S$,

1. $\forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_i, \tilde{x}_i \in V_i\ ...\ \forall\ x_k \in V_k$ where $1 \leq i \leq k$,

$$[u + v](x_1, .., x_i + \tilde{x}_i, .., x_k) = u(x_1, .., x_i + \tilde{x}_i, .., x_k) + v(x_1, .., x_i + \tilde{x}_i, .., x_k)$$

$$= u(x_1, .., x_i, .., x_k) + u(x_1, .., \tilde{x}_i, .., x_k) + v(x_1, .., x_i, .., x_k) + v(x_1, .., \tilde{x}_i, .., x_k)$$
$$= [u + v](x_1, .., x_i, .., x_k) + [u + v](x_1, .., \tilde{x}_i, .., x_k)$$

2. $\forall\, x_1 \in V_1\ \forall\, x_2 \in V_2\ ...\ \forall\, x_i \in V_i\ ...\ \forall\, x_k \in V_k$ where $1 \le i \le k\ \forall\, \alpha \in \mathbb{C}$,

$$\begin{aligned}[u+v](x_1,..,\alpha x_i,..,x_k) &= u(x_1,..,\alpha x_i,..,x_k) + v(x_1,..,\alpha x_i,..,x_k)\\ &= \alpha u(x_1,..,x_i,..,x_k) + \alpha v(x_1,..,x_i,..,x_k)\\ &= \alpha [u+v](x_1,..,x_i,..,x_k)\end{aligned}$$

$$[u+v] \text{ is multi-linear } \implies [u+v] \in S \implies S \text{ is closed under addition}$$

Claim 2 : $\forall\, u \in S\ \forall\, \alpha \in \mathbb{C}\ [\alpha u] \in S$

1. $\forall\, x_1 \in V_1\ \forall\, x_2 \in V_2\ ...\ \forall\, x_i, \tilde{x}_i \in V_i\ ...\ \forall\, x_k \in V_k$ where $1 \le i \le k$,

$$\begin{aligned}[\alpha u](x_1,..,x_i+\tilde{x}_i,..,x_k) &= \alpha u(x_1,..,x_i+\tilde{x}_i,..,x_k)\\ &= \alpha u(x_1,..,x_i,..,x_k) + \alpha u(x_1,..,\tilde{x}_i,..,x_k)\\ &= [\alpha u](x_1,..,x_i,...,x_k) + [\alpha u](x_1,..,x_i,...,x_k)\end{aligned}$$

2. $\forall\, x_1 \in V_1\ \forall\, x_2 \in V_2\ ...\ \forall\, x_i \in V_i\ ...\ \forall\, x_k \in V_k$ where $1 \le i \le k\ \forall\, \beta \in \mathbb{C}$,

$$\begin{aligned}[\alpha u](x_1,..,\beta x_i,..,x_k) = \alpha u(x_1,..,\beta x_i,..,x_k) &= \alpha\beta u(x_1,..,x_i,..,x_k)\\ &= \beta [\alpha u](x_1,..,x_i,..,x_k)\end{aligned}$$

$$[\alpha u] \text{ is multi-linear } \implies [\alpha u] \in S \implies S \text{ is closed under scalar multiplication}$$

□

A multi-linear function $u \in S$ is called a $k-$**tensor** or a multi-linear map on $V_1 \times V_2 \times ... \times V_k$. It is easy to verify that S is a vector space over field $\mathbb{C}$ (We already proved that S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all $k-$tensors is defined as the tensor product space of V_1, V_2, ..., V_k denoted by $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{C})$ or $V_1 \otimes V_2 \otimes ... \otimes V_k$.

2.3.2 Tensor products on vector spaces V_1, V_2, ..., V_k

Definition 2.3.2. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\left(V_i\right) = n_i$ $\forall$ $i \in \{1, 2, ..., k\}$. $\forall$ $u_1 \in \mathcal{L}\left(V_1 \to \mathbb{C}\right)$ $\forall$ $u_2 \in \mathcal{L}\left(V_2 \to \mathbb{C}\right)$ ... $\forall$ $u_k \in \mathcal{L}\left(V_k \to \mathbb{C}\right)$ Define the tensor product of u_1, u_2, ..., u_k as a function $\left[u_1 \otimes u_2 \otimes ... \otimes u_k\right] : V_1 \times V_2 \times ... \times V_k \to \mathbb{C}$ as follows $\forall$ $x_1 \in V_1$ $\forall$ $x_2 \in V_2$ ... $\forall$ $x_k \in V_k$,

$$\boxed{\left[u_1 \otimes u_2 \otimes ... \otimes u_k\right]\left(x_1, x_2, ..., x_k\right) = u_1\left(x_1\right) \cdot u_2\left(x_2\right) \cdot ... \cdot u_k\left(x_k\right) = \prod_{i=1}^{k} u_i\left(x_i\right)}$$

Lemma 2.3.2. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$. $\forall$ $u_1 \in \mathcal{L}\left(V_1 \to \mathbb{C}\right)$ $\forall$ $u_2 \in \mathcal{L}\left(V_2 \to \mathbb{C}\right)$ ... $\forall$ $u_k \in \mathcal{L}\left(V_k \to \mathbb{C}\right)$, $\left[u_1 \otimes u_2 \otimes ... \otimes u_k\right]$ is multi-linear.

Proof. 1. $\forall$ $x_1 \in V_1$ $\forall$ $x_2 \in V_2$ ... $\forall$ $x_i, \tilde{x}_i \in V_i$ ... $\forall$ $x_k \in V_k$ where $1 \leq i \leq k$,

$$\left[u_1 \otimes ... \otimes u_k\right]\left(x_1, ..., x_i + \tilde{x}_i, ..., x_k\right) = u_1\left(x_1\right)...u_i\left(x_i + \tilde{x}_i\right)...u_k\left(x_k\right)$$

$$= u_1\left(x_1\right)...u_i\left(x_i\right)...u_k\left(x_k\right) + u_1\left(x_1\right)...u_i\left(\tilde{x}_i\right)...u_k\left(x_k\right)$$
$$= \left[u_1 \otimes ... \otimes u_k\right]\left(x_1, ..., x_i, ..., x_k\right) + \left[u_1 \otimes ... \otimes u_k\right]\left(x_1, ..., \tilde{x}_i, ..., x_k\right)$$

2. $\forall$ $x_1 \in V_1$ $\forall$ $x_2 \in V_2$ ... $\forall$ $x_i, \tilde{x}_i \in V_i$ ... $\forall$ $x_k \in V_k$ where $1 \leq i \leq k$ $\forall$ $\alpha \in \mathbb{C}$,

$$\begin{aligned}\left[u_1 \otimes ... \otimes u_k\right]\left(x_1, ..., \alpha x_i, ..., x_k\right) &= u_1\left(x_1\right)...u_i\left(\alpha_i x_i\right)...u_k\left(x_k\right) \\ &= \alpha u_1\left(x_1\right)...u_i\left(x_i\right)...u_k\left(x_k\right) \\ &= \alpha\left[u_1 \otimes ... \otimes u_k\right]\left(x_1, x_2, ..., x_k\right)\end{aligned}$$

$$\implies \left[u_1 \otimes ... \otimes u_k\right] \text{ is multi-linear}$$

□

Lemma 2.3.3. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$.

1. $\forall\ u_1 \in \mathcal{L}(V_1 \to \mathbb{C})\ ...\ \forall\ u_i, \tilde{u}_i \in \mathcal{L}(V_i \to \mathbb{C})\ ...\ \forall\ u_k \in \mathcal{L}(V_k \to \mathbb{C})$ where $1 \leq i \leq k$,

$$[u_1 \otimes ... \otimes [u_i + \tilde{u}_i] \otimes ... \otimes u_k] = [u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k] + [u_1 \otimes ... \otimes \tilde{u}_i \otimes ... \otimes u_k]$$

2. $\forall\ u_1 \in \mathcal{L}(V_1 \to \mathbb{C})\ ...\ \forall\ u_i \in \mathcal{L}(V_i \to \mathbb{C})\ ...\ \forall\ u_k \in \mathcal{L}(V_k \to \mathbb{C})$ where $1 \leq i \leq k\ \forall\ \alpha \in \mathbb{C}$,

$$[u_1 \otimes ... \otimes [\alpha u_i] \otimes ... \otimes u_k] = \alpha [u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k]$$

Proof. 1. $\forall\ x_1 \in V_1\ ...\ \forall\ x_k \in V_k$,

$$[u_1 \otimes ... \otimes [u_i + \tilde{u}_i] \otimes ... \otimes u_k](x_1, ..., x_k) = u_1(x_1) ... [u_i + \tilde{u}_i](x_i) ... u_k(x_k)$$

$$= u_1(x_1) ... u_i(x_i) ... u_k(x_k) + u_1(x_1) ... \tilde{u}_i(x_i) ... u_k(x_k)$$
$$= [u_1 \otimes ... u_i ... \otimes u_k](x_1, ..., x_k) + [u_1 \otimes ... \tilde{u}_i ... \otimes u_k](x_1, ..., x_k)$$

2. $\forall\ x_1 \in V_1\ ...\ \forall\ x_k \in V_k\ \forall\ \alpha \in \mathbb{C}$,

$$\begin{aligned}[u_1 \otimes ... \otimes [\alpha u_i] \otimes ... \otimes u_k](x_1, ..., x_k) &= u_1(x_1) ... [\alpha u_i](x_i) ... u_k(x_k) \\ &= \alpha u_1(x_1) ... u_i(x_i) ... u_k(x_k) \\ &= \alpha [u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k](x_1, ..., x_k)\end{aligned}$$

□

Remark :

1. $u_1 \otimes ... \otimes u_k = 0 \iff$ at least one of $u_i = 0$. It is straight forward to verify and left to reader.

2. Note that the tensor products don't have unique representations for instance $\forall\ u_i \in \mathcal{L}(V_i \to \mathbb{C})\ \forall\ \alpha \neq 0 \in \mathbb{C}\ \forall\ i \in \{1, 2, ..., k\}$,

$$u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k = \frac{u_1}{\alpha} \otimes ... \otimes [\alpha u_i] \otimes ... \otimes u_k$$

Illustration :

Consider $V_i = \mathbb{C}^2$ over $\mathbb{C}$ $\forall\ i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Verify that A forms an orthonormal basis of each $V_i = \mathbb{C}^2$. Let $A^* = \{a_1^*, a_2^*\}$. Recall that A^* forms dual basis of V^*. Theorem 2.1.6 implies that $\forall\ u \in V^*\ \forall\ x \in \mathbb{C}^2$,

$$u\,(x) = \overline{{}^{A^*}u} \odot {}^{A}x$$

Let $u_i = \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ $\forall\ i \in \{1, 2, ..., k\}$

$\forall\ x_1, x_2, ..., x_k \in \mathbb{C}^2$,

$$[u_1 \otimes ... \otimes u_k]\,(x_1, ..., x_k) = \prod_{i=1}^{k} u_i\,(x_i) = \prod_{i=1}^{k} \left({}^{A}u_i \odot {}^{A}x_i\right) = \prod_{i=1}^{k} \begin{bmatrix} 1 & -2i \end{bmatrix} \begin{bmatrix} {}^{A}x\,[1] \\ {}^{A}x\,[2] \end{bmatrix}$$

From the linearity of coordinates of vectors $x_i \in V_i$ it is straight forward to conclude that $[u_1 \otimes u_2 \otimes ... \otimes u_k]$ is multi-linear and all the properties in Lemma 3.3.2 hold. Recall that we had an analytic expression to identify $1-$tensor ($\left({}^{A}u\right)^T$) and $2-$ tensor ${}^{A}u \cdot \left({}^{B}v\right)^T$ once a basis is fixed. It is quite clear that such an expression is impossible to get if $k \geq 3$. However, a $k-$tensor can be identified as a $k-$dimensional array computationally $\forall\ k \in \mathbb{N}$.

2.3.3 Basis of $k-$fold tensor product spaces

Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V_i) = n_i$ $\forall\ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be an orthonormal basis of V_1, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be an orthonormal basis of V_2, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be an orthonormal basis of V_k. Define $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a_{1i_1}^* \otimes a_{2i_2}^* \otimes ... \otimes a_{ki_k}^* \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$. Lemma 2.3.2 implies that $\forall\ i_1 \in \{1, 2, ..., n_1\}$ $\forall\ i_2 \in \{1, 2, ..., n_2\}$ ... $\forall\ i_k \in \{1, 2, ..., n_k\}$ $a_{1i_1}^* \otimes a_{2i_2}^* \otimes ... \otimes a_{ki_k}^*$ is multi-linear.

Theorem 2.3.4. $A_1 \otimes A_2 \otimes ... \otimes A_k$ is a basis for vector space $\mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)$

Proof. **Span :**
$\forall\ x_1 \in V_1$ since A_1 is a basis of V_1 there exist unique $\alpha_{1i_1} \in \mathbb{C}$ such that,

$$x_1 = \sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}$$

$\forall\ x_2 \in V_2$ since A_2 is a basis of V_2 there exist unique $\alpha_{2i_2} \in \mathbb{C}$ such that,

$$x_2 = \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}$$

$$...,$$

$\forall\ x_k \in V_k$ since A_k is a basis of V_k there exist unique $\alpha_{ki_k} \in \mathbb{C}$ such that,

$$x_k = \sum_{i_1=1}^{n_k} \alpha_{ki_k} a_{ki_k}$$

$\forall\ u \in \mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)$ since u is multi-linear we get,

$$\begin{aligned} u\left(x_1, x_2, ..., x_k\right) &= u\left(\sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}, \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_3} \alpha_{ki_k} a_{ki_k}\right) \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ki_k} u\left(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}\right) \end{aligned}$$

$\forall\ i_1 \in \{1, ..., n_1\}\ i_2 \in \{1, ..., n_2\}\ ...\ i_k \in \{1, ..., n_k\}$ since $A_1,\ A_2,\ ...,\ A_k$ are orthonormal bases we get,

$$\left[a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}\right]\left(x_1, x_2, ..., x_k\right) = \left(a_{1i_1}, x_1\right)\left(a_{2i_2}, x_2\right) ... \left(a_{ki_k}, x_k\right) = \prod_{j=1}^{k} \alpha_{ji_j}$$

$$\implies u\left(x_1, x_2, ..., x_k\right) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u\left(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}\right)\left[a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}\right]\left(x_1, x_2, ..., x_k\right)$$

$$\implies u = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u\left(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}\right) \left[a_{1i_1}^* \otimes a_{2i_2}^* \otimes ... \otimes a_{ki_k}^*\right]$$

$$\implies A_1 \otimes A_2 \otimes ... \otimes A_k \text{ spans } \mathcal{L}\left(V_1 \times V_2 ... \times V_k \to \mathbb{C}\right)$$

Linear Independence :

Let $\alpha_{i_1,i_2,...,i_k} \in \mathbb{C}\ \forall\ i_1 \in \{1, ..., n_1\}\ i_2 \in \{1, ..., n_2\}\ ...\ i_k \in \{1, ..., n_k\}$. Consider,

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k} \left[a_{1i_1}^* \otimes a_{2i_2}^* \otimes ... \otimes a_{ki_k}^*\right] = 0$$

$\forall\ j_1 \in \{1, ..., n_1\}\ j_2 \in \{1, ..., n_2\}\ ...\ j_k \in \{1, ..., n_k\}$,

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k} \left[a_{1i_1}^* \otimes a_{2i_2}^* \otimes ... \otimes a_{ki_k}^*\right] \left(a_{1j_1}, a_{2j_2}, ..., a_{kj_k}\right) = 0$$

Since A_1, A_2, ..., A_k are orthonormal bases we get,

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k} \left(a_{1i_1}, a_{1j_1}\right) \left(a_{2i_2}, a_{2j_2}\right) ... \left(a_{ki_k}, a_{kj_k}\right) = \alpha_{j_1,j_2,...,j_k} = 0$$

$\implies A_1 \otimes A_2 \otimes ... \otimes A_k$ is a linearly independent set and a basis of $\mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)$

□

Corollary 2.3.5.

$$dim\left(\mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)\right) = dim\left(V_1 \otimes V_2 \otimes ... \otimes V_k\right) = \prod_{i=1}^{k} dim\left(V_i\right)$$

Illustration :

Consider $V_i = \mathbb{C}^2$ over $\mathbb{C}$ $\forall$ $i \in \{1,2,...,k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Verify that A forms orthonormal basis of each $V_i = \mathbb{C}^2$. Let $A^* = \{a_1^*, a_2^*\}$. Recall that A^* is a dual basis of V^*.

$\forall$ $i_1, i_2, ..., i_k \in \{1,2\}$ $\forall$ $x_1, x_2, ..., x_k \in \mathbb{C}^2$,

$$\begin{aligned}
\left[a_{i_1}^* \otimes a_{i_2}^* \otimes ... \otimes a_{i_k}^*\right](x_1, x_2, ..., x_k) &= a_{i_1}^*(x_1) \cdot a_{i_2}^*(x_2) \cdot ... \cdot a_{i_k}^*(x_k) \\
&= \left(\overline{{}^A a_{i_1}} \odot {}^A x_1\right) \cdot \left(\overline{{}^A a_{i_2}} \odot {}^A x_2\right) \cdot ... \cdot \left(\overline{{}^A a_{i_k}} \odot {}^A x_k\right) \\
&= {}^A x_1[i_1] \cdot {}^A x_2[i_2] \cdot ... \cdot {}^A x_k[i_k]
\end{aligned}$$

Let $\underbrace{A \otimes A \otimes ... \otimes A}_{k \text{ times}} = \otimes_{i=1}^k A = \{a_{i_1}^* \otimes a_{i_2}^* \otimes ... \otimes a_{i_k}^* \mid 1 \leq i_1, i_2, ..., i_k \leq 2\}$.

Above Theorem implies that $\otimes_{i=1}^k A$ is a basis of $\mathcal{L}\left(\left(\mathbb{C}^2\right)^k \to \mathbb{R}\right)$,

$$u\left(\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^T, ..., \begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^T\right) = x_{11} \cdot x_{21} ... \cdot x_{k1} = \prod_{i=1}^k x_{i1} \text{ where } \begin{bmatrix} x_{j1} & x_{j2} \end{bmatrix}^T \in \mathbb{C}^2 \ \forall \ j$$

It is straight-forward to verify that u is multi-linear. Since A is a basis of $\mathbb{C}^2$ $\forall$ $\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^T \in \mathbb{C}^2$ there exist unique α_{11}, $\alpha_{12} \in \mathbb{C}$ such that,

$$\begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^T = \alpha_{11} a_1 + \alpha_{12} a_2$$

$\forall$ $\begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^T \in \mathbb{C}^2$ there exist unique α_{21}, $\alpha_{22} \in \mathbb{C}$ such that,

$$\begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^T = \alpha_{21} a_1 + \alpha_{22} a_2$$

$$...,$$

$\forall$ $\begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^T \in \mathbb{C}^2$ there exist unique α_{k1}, $\alpha_{k2} \in \mathbb{C}$ such that,

$$\begin{bmatrix} x_{k1} & x_{k2} \end{bmatrix}^T = \alpha_{k1} a_1 + \alpha_{k2} a_2$$

Since u is multi-linear we get that

$$u\left(\begin{bmatrix}x_{11}\\x_{12}\end{bmatrix},\begin{bmatrix}x_{21}\\x_{22}\end{bmatrix},...,\begin{bmatrix}x_{k1}\\x_{k2}\end{bmatrix}\right)=u\left(\sum_{i_1=1}^{2}\alpha_{1i_1}a_{i_1},\sum_{i_2=1}^{2}\alpha_{2i_2}a_{i_2},...,\sum_{i_k=1}^{2}\alpha_{ki_k}a_{i_k}\right)$$
$$=\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}...\sum_{i_k=1}^{2}\alpha_{1i_1}\alpha_{2i_2}...\alpha_{ki_k}u\left(a_{i_1},a_{i_2},...,a_{i_k}\right)$$

It is quite clear that computing $u\left(a_{i_1},a_{i_2},...,a_{i_k}\right)$ is sufficient to determine the action of u on any $\begin{bmatrix}x_{11} & x_{12}\end{bmatrix}^T,...,\begin{bmatrix}x_{k1} & x_{k2}\end{bmatrix}^T\in\mathbb{C}^2$.

$$\implies u\left(\begin{bmatrix}x_{11}\\x_{12}\end{bmatrix},...,\begin{bmatrix}x_{k1}\\x_{k2}\end{bmatrix}\right)=\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}...\sum_{i_k=1}^{2}u\left(a_{i_1},...,a_{i_k}\right)\left[a^*_{1i_1}\otimes...\otimes a^*_{ki_k}\right]\left(\begin{bmatrix}x_{11}\\x_{12}\end{bmatrix},...,\begin{bmatrix}x_{k1}\\x_{k2}\end{bmatrix}\right)$$

With this illustration, you can observe how $\otimes_{i=1}^{k}A$ works as a basis of $\mathcal{L}\left((\mathbb{C}^2)^k\to\mathbb{C}\right)$ and also note that the values of $u\left(a_{i_1},a_{i_2},...,a_{i_k}\right)$ is sufficient to compute u on any $\begin{bmatrix}x_{11} & x_{12}\end{bmatrix}^T,\begin{bmatrix}x_{21} & x_{22}\end{bmatrix}^T,...,\begin{bmatrix}x_{k1} & x_{k2}\end{bmatrix}^T\in\mathbb{C}^2$.

2.3.4 Basis transformation

Definition 2.3.3. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\left(V_i\right)=n_i\ \forall\ i\in\{1,2,...,k\}$. Let $A_1=\{a_{11},a_{12},...,a_{1n_1}\}$ be an orthonormal basis of V_1, $A_2=\{a_{21},a_{22},...,a_{2n_2}\}$ be an orthonormal basis of V_2, ..., $A_k=\{a_{k1},a_{k2},...,a_{kn_k}\}$ be an orthonormal basis of A_k. Let $A_1\otimes A_2\otimes...\otimes A_k=\{a^*_{1i_1}\otimes a^*_{2i_2}\otimes...\otimes a^*_{ki_k}\mid 1\le i_1\le n_1,1\le i_2\le n_2,...,1\le i_k\le n_k\}$. Theorem 3.3.3 implies that $A_1\otimes A_2\otimes...\otimes A_k$ forms a basis of $\mathcal{L}\left(V_1\times V_2\times...\times V_k\to\mathbb{C}\right)$. $\forall\ u\in\mathcal{L}\left(V_1\times V_2\times...\times V_k\to\mathbb{C}\right)$ we have

$$u=\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k}u\left(a_{1i_1},a_{2i_2},...,a_{ki_k}\right)\left[a_{1i_1}\otimes a_{2i_2}...\otimes a_{ki_k}\right]$$

Define coordinates of the multi-linear function u as follows,

$${}^{A_1\otimes A_2\otimes...\otimes A_k}u\left[i_1,i_2,...,i_k\right]=u\left(a_{1i_1},a_{2i_2},...,a_{ki_k}\right)\qquad\text{where } 1\le i_1\le n_1,...,1\le i_k\le n_k$$

Note :

1. Note that column vector representation is used for $1-$tensors and matrix representation is used for $2-$tensors. Recall from the illustration 2.3.2 that for $k-$tensors such an analytic representation is not possible when $k \geq 3$. But, $k-$tensors can be represented as $k-$dimensional array computationally.

Theorem 2.3.6. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, ..., a_{1n_1}\}$ and $B_1 = \{b_{11}, ..., b_{1n_1}\}$ be any two orthonormal basis of V_1, let $A_2 = \{a_{21}, ..., a_{2n_2}\}$ and $B_2 = \{b_{21}, ..., b_{2n_2}\}$ be any two orthonormal basis of V_2, ..., let $A_k = \{a_{k1}, ..., a_{kn_k}\}$ and $B_k = \{b_{k1}, ..., b_{kn_k}\}$ be any two orthonormal basis of V_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$ and $B_1 \otimes B_2 \otimes ... \otimes B_k = \{b^*_{1j_1} \otimes b^*_{2j_2} \otimes ... \otimes b^*_{kj_k} \mid 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, ..., 1 \leq j_k \leq n_k\}$. Theorem 3.3.3 implies that $A_1 \otimes A_2 \otimes ... \otimes A_k$ and $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}\,(V_1 \times V_2 \times ... \times V_k \to \mathbb{C})$. Let ${}^iM \in \mathbb{C}^{n_i \times n_i}$ be the transformation matrix from A_i to B_i i.e,

$$\begin{bmatrix} a_{i1} & a_{i2} & . & . & . & a_{in_i} \end{bmatrix} = \begin{bmatrix} b_{i1} & b_{i2} & . & . & . & b_{in_i} \end{bmatrix} {}^iM$$

where $1 \leq i \leq k$. $\forall \ u \in \mathcal{L}\,(V_1 \times V_2 \times ... \times V_k \to \mathbb{C})$,

$$\boxed{{}^{B_1 \otimes B_2 \otimes ... \otimes B_k}u\,[j_1, j_2, ..., j_k] = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^1\overline{M}_{j_1 i_1} {}^2\overline{M}_{j_2 i_2} ... {}^k\overline{M}_{j_k i_k} {}^{A_1 \otimes A_2 \otimes ... \otimes A_k}u\,[i_1, i_2, ..., i_k]}$$

where $1 \leq j_1 \leq n_1 1 \leq j_2 \leq n_2 ... 1 \leq j_k \leq n_k$.

Proof. $\forall \ u \in \mathcal{L}\,(V_1 \times V_2 ... \times V_k \to \mathbb{C})$ since $A_1 \otimes A_2 \otimes ... \otimes A_k$ and $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}\,(V_1 \times V_2 \times ... \times V_k \to \mathbb{C})$,

$$\begin{aligned} u &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u\,(a_{1i_1}, a_{2i_2}, ..., a_{ki_k})\,[a_{1i_1} \otimes a_{2i_2} ... \otimes a_{ki_k}] \\ &= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} ... \sum_{j_k=1}^{n_k} u\,(b_{1j_1}, b_{2j_2}, ..., b_{kj_k})\,[b_{1j_1} \otimes b_{2j_2} ... \otimes b_{kj_k}] \end{aligned}$$

Since ${}^{1}M$ is the transformation matrix from basis A_1 to B_1 we get that ${}^{1}M^*$ is the transformation from B_1 to A_1 i.e,

$$\begin{bmatrix} b_{11} & b_{12} & . & . & . & b_{1n_1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n_1} \end{bmatrix} {}^{1}M^*$$

$\forall\ j_1 \in \{1, 2, ..., n_1\}$,

$$b_{1j_1} = \sum_{i_1=1}^{n_1} {}^{1}M^*_{i_1 j_1} a_{1i_1} = \sum_{i_1=1}^{n_1} {}^{1}\overline{M}_{j_1 i_1} a_{1i_1}$$

Since ${}^{2}M$ is the transformation matrix from basis A_2 to B_2 we get that ${}^{2}M^*$ is the transformation from B_2 to A_2 i.e,

$$\begin{bmatrix} b_{21} & b_{22} & . & . & . & b_{2n_2} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & . & . & . & a_{2n_2} \end{bmatrix} {}^{2}M^*$$

$\forall\ j_2 \in \{1, 2, ..., n_2\}$,

$$b_{2j_2} = \sum_{i_2=1}^{n_2} {}^{2}M^*_{i_2 j_2} a_{2i_2} = \sum_{i_2=1}^{n_2} {}^{2}\overline{M}_{j_2 i_2} a_{2i_2}$$

$$...,$$

Since ${}^{k}M$ is the transformation matrix from basis A_k to B_k, we get that ${}^{k}M^*$ is the transformation from B_k to A_k i.e,

$$\begin{bmatrix} b_{k1} & b_{k2} & . & . & . & b_{kn_k} \end{bmatrix} = \begin{bmatrix} a_{k1} & a_{k2} & . & . & . & a_{kn_k} \end{bmatrix} {}^{k}M^*$$

$\forall\ j_k \in \{1, 2, ..., n_k\}$,

$$b_{kj_k} = \sum_{i_k=1}^{n_k} {}^{k}M^*_{i_k j_k} a_{ki_k} = \sum_{i_k=1}^{n_k} {}^{k}\overline{M}_{j_k i_k} a_{ki_k}$$

Since u is multi-linear we get,

$$u\left(b_{1j_1}, b_{2j_2}, ..., b_{kj_k}\right) = u\left(\sum_{i_1=1}^{n_1} {}^{1}\overline{M}_{j_1 i_1} a_{1i_1}, \sum_{i_2=1}^{n_2} {}^{2}\overline{M}_{j_2 i_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_k} {}^{k}\overline{M}_{j_k i_k} a_{ki_k}\right)$$

$$= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \dots \sum_{i_k=1}^{n_k} {}^1\overline{M}_{j_1 i_1} {}^2\overline{M}_{j_2 i_2} \dots {}^k\overline{M}_{j_k i_k} u\left(a_{1i_1}, a_{2i_2}, \dots, a_{ki_k}\right)$$

$$\implies {}^{B_1 \otimes B_2 \otimes \dots \otimes B_k} u\left[j_1, j_2, \dots, j_k\right] = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \dots \sum_{i_k=1}^{n_k} {}^1\overline{M}_{j_1 i_1} {}^2\overline{M}_{j_2 i_2} \dots {}^k\overline{M}_{j_k i_k} {}^{A_1 \otimes A_2 \otimes \dots \otimes A_k} u\left[i_1, i_2, \dots, i_k\right]$$

$\square$

Illustration :

Consider $V_i = \mathbb{C}^2$ over $\mathbb{C}$ $\forall\ i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Verify that A forms orthonormal basis of each $V_i = \mathbb{C}^2$. Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$. Notice that B is the standard orthonormal bases of $\mathbb{C}^2$.
$\forall\ i_1, i_2, ..., i_k \in \{1, 2\}$ $\forall\ x_1, x_2, ..., x_k \in \mathbb{C}^2$,

$$\begin{aligned} \left[a_{i_1}^* \otimes a_{i_2}^* \otimes \dots \otimes a_{i_k}^*\right](x_1, x_2, \dots, x_k) &= a_{i_1}^*(x_1) \cdot a_{i_2}^*(x_2) \cdot \dots \cdot a_{i_k}^*(x_k) \\ &= \left(\overline{{}^A a_{i_1}} \odot {}^A x_1\right) \cdot \left(\overline{{}^A a_{i_2}} \odot {}^A x_2\right) \cdot \dots \cdot \left(\overline{{}^A a_{i_k}} \odot {}^A x_k\right) \\ &= {}^A x_1[i_1] \cdot {}^A x_2[i_2] \cdot \dots \cdot {}^A x_k[i_k] \end{aligned}$$

Let $\underbrace{A \otimes A \otimes \dots \otimes A}_{k \text{ times}} = \otimes_{i=1}^k A = \{a_{i_1}^* \otimes a_{i_2}^* \otimes \dots \otimes a_{i_k}^* \mid 1 \le i_1, i_2, ..., i_k \le 2\}$.
Similarly $\forall\ j_1, j_2, ..., j_k \in \{1, 2\}$ we have

$$\left[b_{j_1}^* \otimes b_{j_2}^* \dots \otimes b_{j_k}^*\right](x_1, x_2, \dots, x_k) = {}^B x_1[j_1] \cdot {}^B x_2[j_2] \cdot \dots \cdot {}^B x_k[j_k]$$

Let $\underbrace{B \otimes B \otimes \dots \otimes B}_{k \text{ times}} = \otimes_{i=1}^k B = \{b_{j_1}^* \otimes b_{j_2}^* \dots \otimes b_{j_k}^* \mid 1 \le j_1, j_2, ..., j_k \le 2\}$. Theorem 3.3.3 implies that $\otimes_{i=1}^k A$ and $\otimes_{j=1}^k B$ form bases of $\mathcal{L}\left((\mathbb{C}^2)^k \to \mathbb{C}\right)$.

Computing the basis transformation matrix from basis A to B

$$a_1 = \frac{1}{\sqrt{2}} b_1 + \frac{i}{\sqrt{2}} b_2 \qquad a_2 = \frac{1}{\sqrt{2}} b_1 - \frac{i}{\sqrt{2}} b_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \implies M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$$

We get that M is the transformation matrix from basis A to B of $\mathbb{C}^2$ which implies that M^* is the transformation matrix from basis B to A i.e,

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$\implies b_1 = \frac{1}{\sqrt{2}}a_1 + \frac{1}{\sqrt{2}}a_2 \qquad b_2 = -\frac{i}{\sqrt{2}}a_1 + \frac{i}{\sqrt{2}}a_2$$

Notice that each ${}^{i}M = M$ since each $V_i = \mathbb{C}^2$.
$\forall\ u \in \mathcal{L}\left((\mathbb{C}^2)^k \to \mathbb{C}\right)\ \forall\ j_1, j_2, ..., j_k \in \{1, 2\}$ we get

$$\begin{aligned} u\left(b_{1j_1}, b_{2j_2}, ..., b_{kj_k}\right) &= u\left(\sum_{i=1}^{2} M^*_{i_1j_1}a_{i_1}, \sum_{i_2=1}^{2} M^*_{i_2j_2}a_{i_2}, ..., \sum_{i_k=1}^{2} M^*_{i_kj_k}a_{i_k}\right) \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}...\sum_{i_k=1}^{2} \overline{M}_{j_1i_1}\overline{M}_{j_2i_2}...\overline{M}_{j_ki_k} u\left(a_{i_1}, a_{i_2}, ..., a_{i_k}\right) \end{aligned}$$

$$\implies {}^{B_1\otimes B_2\otimes...\otimes B_k}u\left[j_1, j_2, ..., j_k\right] = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}...\sum_{i_k=1}^{2} \overline{M}_{j_1i_1}\overline{M}_{j_2i_2}...\overline{M}_{j_ki_k}\ {}^{A_1\otimes A_2\otimes...\otimes A_k}u\left[i_1, i_2, ..., i_k\right]$$

2.3.5 Invariance of computation of k-tensors under any orthonormal basis transformations

Theorem 2.3.7. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\left(V_i\right) = n_i\ \forall\ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, ..., a_{1n_1}\}$ be any orthonormal basis of V_1, let $A_2 = \{a_{21}, ..., a_{2n_2}\}$ be any orthonormal basis of V_2, ..., let $A_k = \{a_{k1}, ..., a_{kn_k}\}$ be any orthonormal basis of V_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k} \mid 1 \le i_1 \le n_1, 1 \le i_2 \le n_2, ..., 1 \le i_k \le n_k\}$. Theorem 3.3.3 implies that $A_1 \otimes A_2 \otimes ... \otimes A_k$ form basis of $\mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)\ \forall\ x_1 \in V_1$ $\forall\ x_2 \in V_2$ ... $\forall\ x_k \in V_k$,

$$\boxed{u\left(x_1, x_2, ..., x_k\right) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}...\sum_{i_k=1}^{n_k} {}^{A_1\otimes A_2\otimes...\otimes A_k}u\left[i_1, i_2, ..., i_k\right]\ {}^{A_1}x_1\left[i_1\right]\ {}^{A_2}x_2\left[i_2\right] ...\ {}^{A_k}x_k\left[i_k\right]}$$

Proof. $\forall\ x_1 \in V_1$ since A_1 is a basis of V_1 there exist unique $\alpha_{1i_1} \in \mathbb{C}$ such that,

$$x_1 = \sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}$$

$\forall\ x_2 \in V_2$ since A_2 is a basis of V_2 there exist unique $\alpha_{2i_2} \in \mathbb{C}$ such that,

$$x_2 = \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}$$

$$...,$$

$\forall\ x_k \in V_k$ since A_k is a basis of V_k there exist unique $\alpha_{ki_k} \in \mathbb{C}$ such that,

$$x_k = \sum_{i_k=1}^{n_k} \alpha_{ki_k} a_{ki_k}$$

$\forall\ u \in \mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)$ we have,

$$\begin{aligned} u\left(x_1, x_2, ..., x_k\right) &= u\left(\sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}, \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_k} \alpha_{ki_k} a_{ki_k}\right) \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ki_k} u\left(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}\right) \end{aligned}$$

Since A_1, A_2, ..., A_k are bases of V_1, V_2, ..., V_k respectively we get that,

$$\alpha_{1i_1} = {}^{A_1}x_1\left[i_1\right] \qquad \alpha_{2i_2} = {}^{A_2}x_2\left[i_2\right] \qquad ... \qquad \alpha_{ki_k} = {}^{A_k}x_k\left[i_k\right]$$

Since $A_1 \otimes A_2 \otimes ... \otimes A_k$ forms basis of $\mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}\right)$, we get

$$u\left(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}\right) = {}^{A_1 \otimes A_2 ... \otimes A_k}u\left[i_1, i_2, ..., i_k\right]$$

$$\implies u\left(x_1, x_2, ..., x_k\right) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{A_1 \otimes A_2 ... \otimes A_k}u\left[i_1, i_2, ..., i_k\right] {}^{A_1}x_1\left[i_1\right] {}^{A_2}x_2\left[i_2\right] ... {}^{A_k}x_k\left[i_k\right]$$

□

Remark :

1. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V_i) = n_i\ \forall\ i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, ..., a_{1n_1}\}$ and $B_1 = \{b_{11}, ..., b_{1n_1}\}$ be any two orthonormal basis of V_1, let $A_2 = \{a_{21}, ..., a_{2n_2}\}$ and $B_2 = \{b_{21}, ..., b_{2n_2}\}$ be any two orthonormal basis of V_2, ..., let $A_k = \{a_{k1}, ..., a_{kn_k}\}$ and $B_k = \{b_{k1}, ..., b_{kn_k}\}$ be any two orthonormal basis of V_k. Let $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$ and $B_1 \otimes B_2 \otimes ... \otimes B_k = \{b^*_{1j_1} \otimes b^*_{2j_2} \otimes ... \otimes b^*_{kj_k} \mid 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, ..., 1 \leq j_k \leq n_k\}$. Theorem 3.3.3 implies that $A_1 \otimes A_2 \otimes ... \otimes A_k$ and $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}\,(V_1 \times V_2 \times ... \times V_k \to \mathbb{C})$. $\forall\ x_1 \in V_1$ $\forall\ x_2 \in V_2$ ... $\forall\ x_k \in V_k$ we get,

$$\begin{aligned} u\,(x_1, x_2, ..., x_k) &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u\,[i_1, i_2, ..., i_k]\ {}^{A_1}x_1\,[i_1]\ {}^{A_2}x_2\,[i_2] ...\ {}^{A_k}x_k\,[i_k] \\ &= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} ... \sum_{j_k=1}^{n_k} {}^{B_1 \otimes B_2 \otimes ... \otimes B_k} u\,[j_1, j_2, ..., j_k]\ {}^{B_1}x_1\,[j_1]\ {}^{B_2}x_2\,[j_2] ...\ {}^{B_k}x_k\,[j_k] \end{aligned}$$

2. It is easy to observe that $\forall\ (x_1, x_2, ..., x_k) \in V_1 \times V_2 \times ... \times V_k$ $u\,(x_1, x_2, ..., x_k)$ can be completely determined by the coordinates of u i.e, ${}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u\,[i_1, i_2, ..., i_k]$. Hence if we fix computations with respect to orthonormal bases A_1, A_2, ..., A_k of V_1, V_2, ..., V_k respectively we can identify u with its coordinates ${}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u\,[i_1, i_2, ..., i_k]$.

3. Let $dim\,(V_i) = n_i$ then $\forall\ u \in \mathcal{L}\,(V_1 \times V_2 \times ... \times V_k \to \mathbb{R})$,

$$ {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u\,[i_1, i_2, ..., i_k] = u\,(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) $$

   Hence $k-$fold tensor product space $\mathcal{L}\,(V_1 \times V_2 \times ... \times V_k \to \mathbb{C})$ is isomorphic to $\mathbb{C}^{n_1 \times n_2 \times ... \times n_k}$. (It is straight-forward to verify and is left to the reader. For the proof technique you may refer Lemma 1.1.9)

Illustration :

Consider $V_i = \mathbb{C}^2$ over $\mathbb{C}$ $\forall$ $i \in \{1, 2, ..., k\}$ with standard dot product as inner product. Let $A = \{a_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}^T, a_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}^T\}$. Verify that A forms orthonormal basis of each $V_i = \mathbb{C}^2$. Let $B = \{b_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$. Notice that B is the standard orthonormal bases of $\mathbb{C}^2$.

$\forall$ $i_1, i_2, ..., i_k \in \{1, 2\}$ $\forall$ $x_1, x_2, ..., x_k \in \mathbb{C}^2$,

$$\begin{aligned} \left[a^*_{i_1} \otimes a^*_{i_2} \otimes ... \otimes a^*_{i_k}\right](x_1, x_2, ..., x_k) &= a^*_{i_1}(x_1) \cdot a^*_{i_2}(x_2) \cdot ... \cdot a^*_{i_k}(x_k) \\ &= \left(\overline{{}^A a_{i_1}} \odot {}^A x_1\right) \cdot \left(\overline{{}^A a_{i_2}} \odot {}^A x_2\right) \cdot ... \cdot \left(\overline{{}^A a_{i_k}} \odot {}^A x_k\right) \\ &= {}^A x_1[i_1] \cdot {}^A x_2[i_2] \cdot ... \cdot {}^A x_k[i_k] \end{aligned}$$

Let $\underbrace{A \otimes A \otimes ... \otimes A}_{k \text{ times}} = \otimes^k_{i=1} A = \{a^*_{i_1} \otimes a^*_{i_2} \otimes ... \otimes a^*_{i_k} \mid 1 \leq i_1, i_2, ..., i_k \leq 2\}$.
Similarly $\forall$ $j_1, j_2, ..., j_k \in \{1, 2\}$ we have

$$\left[b^*_{j_1} \otimes b^*_{j_2} ... \otimes b^*_{j_k}\right](x_1, x_2, ..., x_k) = {}^B x_1[j_1] \cdot {}^B x_2[j_2] \cdot ... \cdot {}^B x_k[j_k]$$

Let $\underbrace{B \otimes B \otimes ... \otimes B}_{k \text{ times}} = \otimes^k_{i=1} B = \{b^*_{j_1} \otimes b^*_{j_2} ... \otimes b^*_{j_k} \mid 1 \leq j_1, j_2, ..., j_k \leq 2\}$. Theorem 3.3.3 implies that $\otimes^k_{i=1} A$ and $\otimes^k_{j=1} B$ form bases of $\mathcal{L}\left((\mathbb{C}^2)^k \to \mathbb{C}\right)$. In the previous illustration, we have already shown that the basis transformation matrix M from A to B is

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$$

$\forall$ $x \in \mathbb{C}^2$,

$${}^B x = M \cdot {}^A x$$

In this illustration we show that $\forall$ $u_1, u_2, ..., u_k \in \mathbb{C}^2$, $[u_1 \otimes u_2 \otimes ... \otimes u_k]$ has the same value irrespective of the choice of orthonormal basis and this is sufficient to claim that any $v \in \mathcal{L}\left((\mathbb{C}^2)^k \to \mathbb{C}\right)$ has the same value irrespective of the choice of orthonormal basis since any $v \in \mathcal{L}\left((\mathbb{C}^2)^k \to \mathbb{C}\right)$ can be written as a linear combination of tensor products in $\otimes_{i=1}^{k} A$.

$$\begin{aligned}
\left[u_1 \otimes u_2 \otimes ... \otimes u_k\right](x_1, x_2, ..., x_k) &= u_1(x_1) \cdot u_2(x_2) \cdot ... \cdot u_k(x_k) \\
&= \left(\overline{{}^{B}u_1} \odot {}^{B}x_1\right) \cdot \left(\overline{{}^{B}u_2} \odot {}^{B}x_2\right) \cdot ... \cdot \left(\overline{{}^{B}u_k} \odot {}^{B}x_k\right) \\
&= \prod_{i=1}^{k} \left(\overline{{}^{B}u_i} \odot {}^{B}x_i\right) = \prod_{i=1}^{k} \left(\overline{\overline{M} \cdot {}^{A}u_i}\right) \odot \left(M \cdot {}^{A}x_i\right) \\
&= \prod_{i=1}^{k} \left(\left({}^{A}u_i\right)^{T} \cdot {}^{A}x_i\right) = \prod_{i=1}^{k} \left(\overline{{}^{A}u_i} \odot {}^{A}x_i\right)
\end{aligned}$$

2.3.6 Inner products on k-fold tensor product spaces

In this section we define a function $() : (V_1 \otimes V_2 \otimes ... \otimes V_k) \times (V_1 \otimes V_2 \otimes ... \otimes V_k) \to \mathbb{C}$ in terms of inner products defined on V_1^*, V_2^*, ..., V_k^* and prove that this function is an inner product.

Definition 2.3.4. Let V_1, V_2, ..., V_k be any finite dimensional inner product space where $\dim(V_i) = n_i \ \forall\ i \in \{1, 2, ..., k\}$. $\forall\ u_{11}, ..., u_{1r}, v_{11}, ..., v_{1s} \in V_1^*$ $\forall$ $u_{21}, ..., u_{2r}, v_{21}, ..., v_{2s} \in V_2^*$ ... $\forall\ u_{k1}, ..., u_{kr}, v_{k1}, ..., v_{ks} \in V_k^*$ $\forall\ \alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s \in \mathbb{C}$ Define the following function $() : (V_1 \otimes V_2 \otimes ... \otimes V_k) \times (V_1 \otimes V_2 \otimes ... \otimes V_k) \to \mathbb{C}$,

$$\boxed{\left(\sum_{i=1}^{r} \alpha_i \left[u_{1i} \otimes ... \otimes u_{ki}\right], \sum_{j=1}^{s} \beta_j \left[v_{1i} \otimes ... \otimes v_{ki}\right]\right) = \sum_{i=1}^{r} \sum_{j=1}^{s} \overline{\alpha}_i \beta_j \prod_{l=1}^{k} \left(u_{li}, v_{lj}\right)_l} \tag{2.5}$$

Note that for any $l \in \{1, 2, ..., k\}$ $()_l$ is an inner product on V_l^*. In the subsequent analysis we drop the subscripts since we believe that the context of usage shall be clear.

In this lemma it is shown that how above definition can be used to compute $(u, v)\ \forall\ u, v \in V_1 \otimes V_2 \otimes ... \otimes V_k$.

Lemma 2.3.8. Let V_1, V_2, ..., V_k be any finite dimensional inner product spaces where $\dim(V_i) = n_i \ \forall\ i \in \{1, 2, ..., k\}$. Let $A_{11} = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be any orthonormal basis of V_1, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be any orthonormal basis of V_2, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be any orthonormal basis of V_k. $\forall\ u, v \in V_1 \otimes V_2 \otimes ... \otimes V_k$

since $A_1 \otimes A_2 \otimes ... \otimes A_k$ is a basis of $V_1 \otimes V_2 \otimes ... \otimes V_k$ there exist $\alpha_{i_1,i_2,...,i_k}, \beta_{j_1,j_2,...,j_k} \in \mathbb{C}$ such that,

$$u = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} \left[a^*_{1i_1} \otimes ... \otimes a^*_{ki_k}\right] \qquad v = \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{j_1,...,j_k} \left[a^*_{1j_1} \otimes ... \otimes a^*_{kj_k}\right]$$

$$\boxed{(u,v) = \sum_{i_1=1}^{n_1} ... \sum_{i_k}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \beta_{j_1,...,j_k} \left(a^*_{1i_1}, a^*_{1j_1}\right) ... \left(a^*_{ki_k}, a^*_{kj_k}\right)} \tag{2.6}$$

Proof. Proof is straight forward and left to reader (use equation 2.5). □

Remark :

1. The existence of inner products on dual space i.e, $()_1$, $()_2$, ..., $()_k$ is already shown in section 2.1.6. Also It is already shown that with respect to the inner product defined in section 2.1.6 A_1^*, A_2^*, ..., A_k^* are orthonormal bases which implies that,

$$(u,v) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \beta_{i_1,...,i_k} \tag{2.7}$$

2. Note that in the subsequent analysis we consider arbitrary inner products on V_1^*, V_2^*, ..., V_k^* in order to make the theory more general. Hence equation 2.6 is used instead of equation 2.7.

Lemma 2.3.9. $\forall\, u, v \in V_1 \otimes V_2 \otimes ... \otimes V_k$,

$$(u,v) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \beta_{j_1,...,j_k} \left(a^*_{1i_1}, a^*_{1j_1}\right) ... \left(a^*_{ki_k}, a^*_{kj_k}\right) \quad \text{is well-defined}$$

Proof. Let $B_{11} = \{b_{11}, b_{12}, ..., b_{1n_1}\}$ be any another orthonormal basis of V_1, $B_2 = \{b_{21}, b_{22}, ..., b_{2n_2}\}$ be any another orthonormal basis of V_2, ..., $B_k = \{b_{k1}, b_{k2}, ..., b_{kn_k}\}$ be any another orthonormal basis of V_k. Since $B_1 \otimes B_2 \otimes ... \otimes B_k$ is a basis of $V_1 \otimes V_2 \otimes ... \otimes B_k$ there exist $\gamma_{p_1,p_2,...,p_k}, \delta_{q_1,q_2,...,q_k} \in \mathbb{C}$ such that,

$$u = \sum_{p_1=1}^{n_1} ... \sum_{p_k=1}^{n_k} \gamma_{p_1,...,p_k} \left[b^*_{1p_1} \otimes ... \otimes b^*_{kp_k}\right] \qquad v = \sum_{q_1=1}^{n_1} ... \sum_{q_k=1}^{n_k} \delta_{q_1,...,q_k} \left[b^*_{1q_1} \otimes ... \otimes b^*_{kq_k}\right]$$

Inner product of u and v using basis B_1^*, B_2^*, ..., B_k^* is

$$(u,v) = \sum_{p_1=1}^{n_1} \dots \sum_{p_k=1}^{n_k} \sum_{q_1=1}^{n_1} \dots \sum_{q_k=1}^{n_k} \overline{\gamma}_{p_1,\dots,p_k} \delta_{q_1,\dots,q_k} \left(b_{1p_1}^*, b_{1q_1}^*\right) \dots \left(b_{kp_k}^*, b_{kq_k}^*\right)$$

To claim (u,v) is well-defined it is enough to show that

$$\begin{aligned} &\sum_{i_1=1}^{n_1} \dots \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} \dots \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,\dots,i_k} \beta_{j_1,\dots,j_k} \left(a_{1i_1}^*, a_{1j_1}^*\right) \dots \left(a_{ki_k}^*, a_{kj_k}^*\right) \\ &= \sum_{p_1=1}^{n_1} \dots \sum_{p_k=1}^{n_k} \sum_{q_1=1}^{n_1} \dots \sum_{q_k=1}^{n_k} \overline{\gamma}_{p_1,\dots,p_k} \delta_{q_1,\dots,q_k} \left(b_{1p_1}^*, b_{1q_1}^*\right) \dots \left(b_{kp_k}^*, b_{kq_k}^*\right) \end{aligned}$$

Let ${}^1M \in \mathbb{C}^{n_1 \times n_1}$ be the transformation matrix from basis A_1 to B_1. Theorem 2.1.5 implies that,

$$\begin{bmatrix} a_{11}^* & a_{12}^* & . & . & a_{1n_1}^* \end{bmatrix} = \begin{bmatrix} b_{11}^* & b_{12}^* & . & . & b_{1n_1}^* \end{bmatrix} {}^1\overline{M}$$

$$\implies a_{1i_1}^* = \sum_{p_1=1}^{n_1} {}^1\overline{M}_{p_1 i_1} b_{1p_1}^* \text{ and } a_{1j_1}^* = \sum_{q_1=1}^{n_1} {}^1\overline{M}_{q_1 j_1} b_{1q_1}^*$$

...,

Let ${}^kM \in \mathbb{C}^{n_k \times n_k}$ be the transformation matrix from basis A_k to B_k. Theorem 2.1.5 implies that,

$$\begin{bmatrix} a_{k1}^* & a_{k2}^* & . & . & a_{kn_k}^* \end{bmatrix} = \begin{bmatrix} b_{k1}^* & b_{k2}^* & . & . & b_{kn_k}^* \end{bmatrix} {}^k\overline{M}$$

$$\implies a_{ki_k}^* = \sum_{p_k=1}^{n_k} {}^k\overline{M}_{p_k i_k} b_{kp_k}^* \text{ and } a_{kj_k}^* = \sum_{q_k=1}^{n_k} {}^k\overline{M}_{q_k j_k} b_{kq_k}^*$$

Above two equations imply that,

$$\sum_{i_1=1}^{n_1}\dots\sum_{i_k=1}^{n_k}\sum_{j_1=1}^{n_1}\dots\sum_{j_k=1}^{n_k}\overline{\alpha}_{i_1,\dots,i_k}\beta_{j_1,\dots,j_k}\left(a^*_{1i_1},a^*_{1j_1}\right)\dots\left(a^*_{ki_k},a^*_{kj_k}\right)$$
$$=\sum_{i_1=1}^{n_1}\dots\sum_{i_k=1}^{n_k}\sum_{j_1=1}^{n_1}\dots\sum_{j_k=1}^{n_k}\sum_{p_1=1}^{n_1}\dots\sum_{p_k=1}^{n_k}\sum_{q_1=1}^{n_1}\dots\sum_{q_k=1}^{n_k}\overline{\alpha}_{i_1,\dots,i_k}\beta_{j_1,\dots,j_k}{}^1M_{p_1i_1}\dots{}^kM_{p_ki_k}$$
$${}^1\overline{M}_{q_1j_1}\dots{}^k\overline{M}_{q_kj_k}\left(b^*_{1p_1},b^*_{1q_1}\right)\dots\left(b^*_{kp_k},b^*_{kq_k}\right)$$
$$=\sum_{p_1=1}^{n_1}\dots\sum_{p_k=1}^{n_k}\sum_{q_1=1}^{n_1}\dots\sum_{q_k=1}^{n_k}\overline{\left(\sum_{i_1=1}^{n_1}\dots\sum_{i_k=1}^{n_k}\alpha_{i_1,i_2,\dots,i_k}{}^1\overline{M}_{p_1i_1}\dots{}^k\overline{M}_{p_ki_k}\right)}$$
$$\left(\sum_{j_1=1}^{n_1}\dots\sum_{j_k=1}^{n_k}\beta_{j_1,\dots,j_k}{}^1\overline{M}_{q_1j_1}\dots{}^k\overline{M}_{q_kj_k}\right)\left(b^*_{1p_1},b^*_{1q_1}\right)\dots\left(b^*_{kp_k},b^*_{kq_k}\right)$$

Note that

$${}^{A_1\otimes A_2\otimes\dots\otimes A_k}u\left[i_1,i_2,\dots,i_k\right]=\alpha_{i_1,i_2,\dots,i_k}\qquad {}^{B_1\otimes B_2\otimes\dots\otimes B_k}u\left[p_1,p_2,\dots,p_k\right]=\gamma_{p_1,\dots,p_k}$$

$${}^{A_1\otimes A_2\otimes\dots\otimes A_k}v\left[j_1,j_2,\dots,j_k\right]=\beta_{j_1,j_2,\dots,j_k}\qquad {}^{B_1\otimes B_2\otimes\dots\otimes B_k}v\left[q_1,q_2,\dots,q_k\right]=\delta_{q_1,\dots,q_k}$$

From theorem 2.3.6 we get,

$${}^{B_1\otimes B_2\otimes\dots\otimes B_k}u\left[p_1,p_2,\dots,p_k\right]=\sum_{i_1=1}^{n_1}\dots\sum_{i_k=1}^{n_k}{}^1\overline{M}_{p_1i_1}\dots{}^k\overline{M}_{p_ki_k}{}^{A_1\otimes\dots\otimes A_k}u\left[i_1,\dots,i_k\right]$$

$${}^{B_1\otimes B_2\otimes\dots\otimes B_k}v\left[q_1,q_2,\dots,q_k\right]=\sum_{j_1=1}^{n_1}\dots\sum_{j_k=1}^{n_k}{}^1\overline{M}_{q_1j_1}\dots{}^k\overline{M}_{q_kj_k}{}^{A_1\otimes\dots\otimes A_k}v\left[j_1,\dots,j_k\right]$$

$$\implies\quad \begin{aligned}&\sum_{i_1=1}^{n_1}\dots\sum_{i_k=1}^{n_k}\sum_{j_1=1}^{n_1}\dots\sum_{j_k=1}^{n_k}\overline{\alpha}_{i_1,\dots,i_k}\beta_{j_1,\dots,j_k}\left(a^*_{1i_1},a^*_{1j_1}\right)\dots\left(a^*_{ki_k},a^*_{kj_k}\right)\\&=\sum_{p_1=1}^{n_1}\dots\sum_{p_k=1}^{n_k}\sum_{q_1=1}^{n_1}\dots\sum_{q_k=1}^{n_k}\overline{\gamma}_{p_1,\dots,p_k}\delta_{q_1,\dots,q_k}\left(b^*_{1p_1},b^*_{1q_1}\right)\dots\left(b^*_{kp_k},b^*_{kq_k}\right)\end{aligned}$$

□

Lemma 2.3.10. $\forall\, u, v \in V_1 \otimes V_2 \otimes ... \otimes V_k$,

$$(u, v) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \beta_{j_1,...,j_k} \left(a_{1i_1}^*, a_{1j_1}^*\right) ... \left(a_{ki_k}^*, a_{kj_k}^*\right) \text{ is an inner product}$$

Proof. **Linearity :** $\forall\, u, v, w \in V_1 \otimes V_2 \otimes ... \otimes V_k \;\forall\, \delta \in \mathbb{C}$,

$$\begin{aligned}(u, v + w) &= \left(\sum_{i_1=1}^{n_1} .. \sum_{i_k=1}^{n_k} \alpha_{i_1,..,i_k} a_{1i_1}^* \otimes .. \otimes a_{ki_k}^*, \sum_{j_1=1}^{n_1} .. \sum_{j_k=1}^{n_k} \left(\beta_{j_1,..,j_k} + \gamma_{j_1..,j_k}\right) a_{1j_1}^* \otimes .. \otimes a_{kj_k}^* \right) \\ &= \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \left(\beta_{j_1,...,j_k} + \gamma_{j_1,...,j_k}\right) \left(a_{1i_1}^*, a_{1j_1}^*\right) ... \left(a_{ki_k}^*, a_{kj_k}^*\right) \\ &= (u, v) + (u, w)\end{aligned}$$

$$\begin{aligned}(u, \delta v) &= \left(\sum_{i_1=1}^{n_1} .. \sum_{i_k=1}^{n_k} \alpha_{i_1,..,i_k} a_{1i_1}^* \otimes .. \otimes a_{ki_k}^*, \sum_{j_1=1}^{n_1} .. \sum_{j_k=1}^{n_k} \left(\delta\beta_{j_1,..,j_k}\right) a_{1j_1}^* \otimes .. \otimes a_{kj_k}^* \right) \\ &= \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \delta\beta_{j_1,...,j_k} \left(a_{1i_1}^*, a_{1j_1}^*\right) ... \left(a_{ki_k}^*, a_{kj_k}^*\right) \\ &= \delta\,(u, v)\end{aligned}$$

Conjugate Symmetry : $\forall\, u, v \in V_1 \otimes V_2 \otimes ... \otimes V_k$,

$$\begin{aligned}(u, v) &= \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \overline{\alpha}_{i_1,...,i_k} \left(\beta_{j_1,...,j_k} + \gamma_{j_1,...,j_k}\right) \left(a_{1i_1}^*, a_{1j_1}^*\right) ... \left(a_{ki_k}^*, a_{kj_k}^*\right) \\ &= \overline{\sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \alpha_{i_1,...,i_k} \overline{\beta}_{j_1,...,j_k} \left(a_{1j_1}^*, a_{1i_1}^*\right) ... \left(a_{kj_k}^*, a_{ki_k}^*\right)} \\ &= \overline{(v, u)}\end{aligned}$$

Positive Definiteness : Since each V_i^* over $\mathbb{C}$ is an inner product space using Gram Schmidt process there exist an orthonormal basis for V_i^*. Without loss of generality assume $A_1^*, A_2^*, ..., A_k^*$ form orthonormal bases of $V_1^*, V_2^*, ..., V_k^*$ respectively.

$$(u, u) = 0 \iff \left(\sum_{i_1=1}^{n_1} .. \sum_{i_k=1}^{n_k} \alpha_{i_1,..,i_k} \left[a_{1i_1}^* \otimes .. \otimes a_{ki_k}^*\right], \sum_{j_1=1}^{n_1} .. \sum_{j_k=1}^{n_k} \alpha_{j_1,..,j_k} \left[a_{1j_1}^* \otimes .. \otimes a_{kj_k}^*\right]\right) = 0$$

$$\iff \sum_{i_1=1}^{n_1} \dots \sum_{i_k=1}^{n_k} \overline{\alpha}_{i_1,\dots,i_k} \alpha_{i_1,\dots,i_k} = \sum_{i_1=1}^{n_1} \dots \sum_{i_k=1}^{n_k} |\alpha_{i_1,\dots,i_k}|^2 = 0$$
$$\iff u = 0$$

□

We end this section showing how inner products can be used in giving an alternate proof for the linear independence of the set $A_1 \otimes A_2 \otimes ... \otimes A_k$. Consider the inner product on dual space defined as in section 2.1.6. $\forall\ i_1, j_1 \in \{1, 2, ..., n_1\}$ ... $\forall$ $i_k, j_k \in \{1, 2, ..., n_k\}$,

$$\left(a^*_{1i_1} \otimes .. \otimes a^*_{ki_k}, a^*_{1j_1} \otimes .. \otimes a^*_{kj_k}\right) = \left(a^*_{1i_1}, a^*_{1j_1}\right) .. \left(a^*_{ki_k}, a^*_{kj_k}\right) = 1 \text{ if } (i_1, .., i_k) = (j_1, .., j_k)$$
$$= 0 \quad \text{otherwise}$$

From lemma 1.1.7 it is straight forward to verify linear independence of the set $A_1 \otimes A_2 \otimes ... \otimes A_k$.

2.3.7 Linear operators on k-fold tensor product spaces

Definition 2.3.5. Let $\mathcal{L}(V_1 \otimes ... \otimes V_k)$ denote the set of all linear operators over the tensor product space of V_1, V_2, ..., V_k $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{C}) = V_1 \otimes V_2 \otimes ... \otimes V_k$. Define addition and scalar multiplication on the set $\mathcal{L}(V_1 \otimes ... \otimes V_k)$ as follows $\forall\ T, W \in \mathcal{L}(V_1 \otimes ... \otimes V_k)$ $\forall\ u \in V_1 \otimes ... \otimes V_k$ $\forall\ \alpha \in \mathbb{C}$,

$$[T + W]\, u = T(u) + W(u)$$

$$[\alpha T]\, u = \alpha \cdot T(u)$$

Remark :

1. It is straight forward to verify that $\mathcal{L}(V_1 \otimes ... \otimes V_k)$ is a vector space over $\mathbb{C}$ and is left to reader (refer lemma 2.3.1).

2. Note that $\mathcal{L}(V_1 \otimes V_2 \otimes ... \otimes V_k)$ is also called tensor product space of operators on the dual space $V_1 \otimes V_2 \otimes ... \otimes V_k$.

Next we define tensor product of k operators and show that any operator on $V_1 \otimes ... \otimes V_k$ can be expressed in terms of tensor products.

Definition 2.3.6. Let V_1, V_2, ..., V_k be any finite dimensional inner product spaces over field $\mathbb{C}$ where $\dim(V_i) = n_i \ \forall \ i \in \{1, 2, ..., k\}$. Let T_1 be an operator on V_1^*, ..., T_k be an operator on V_k^*. Define the tensor product of T_1, ..., T_k on $V_1 \otimes V_2 \otimes ... \otimes V_k$ i.e, $T_1 \otimes ... \otimes T_k : (V_1 \otimes ... \otimes V_k) \to (V_1 \otimes ... \otimes V_k) \ \forall \ x_{11}, ..., x_{1l} \in V_1^*$ ... $\forall \ x_{k1}, ..., x_{kl} \in V_k^* \ \forall \ \alpha_1, \alpha_2, ..., \alpha_l \in \mathbb{C}$,

$$\boxed{[T_1 \otimes ... \otimes T_k] \left(\sum_{i=1}^{l} \alpha_i x_{1i} \otimes x_{2i} \otimes ... \otimes x_{ki} \right) = \sum_{i=1}^{l} \alpha_i \left[T_1 \left(x_{1i}\right)\right] \otimes \left[T_2 \left(x_{2i}\right)\right] \otimes ... \otimes \left[T_k \left(x_{ki}\right)\right]} \tag{2.8}$$

In this lemma it is shown that how above definition can be used to compute $[T_1 \otimes ... \otimes T_k](x) \ \forall \ x \in V_1 \otimes V_2 \otimes ... \otimes V_k$.

Lemma 2.3.11. Let $A_1 = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be any orthonormal basis of V_1 and $A_1^* = \{a_{11}^*, a_{12}^*, ..., a_{1n_1}^*\}$ be the corresponding dual basis, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be any orthonormal basis of V_2 and $A_2^* = \{a_{21}^*, a_{22}^*, ..., a_{2n_2}^*\}$ be the corresponding dual basis, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be any orthonormal basis of V_k and $A_k^* = \{a_{k1}^*, a_{k2}^*, ..., a_{kn_k}^*\}$ be the corresponding dual basis. $\forall \ x \in V_1 \otimes V_2 \otimes ... \otimes V_k$ since $A_1 \otimes A_2 \otimes ... \otimes A_k$ is a basis of $V_1 \otimes V_2 \otimes ... \otimes V_k$ there exist unique $\alpha_{i_1, i_2, ..., i_k} \in \mathbb{C}$ such that,

$$x = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1, i_2, ..., i_k} a_{1i_1}^* \otimes a_{2i_2}^* \otimes ... \otimes a_{ki_k}^*$$

$\forall \ T_1 \in \mathcal{L}(V_1^*) \ \forall \ T_2 \in \mathcal{L}(V_2^*)$ ... $\forall \ T_k \in \mathcal{L}(V_k^*)$,

$$[T_1 \otimes ... \otimes T_k](x) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1, ..., i_k} \left[T\left(a_{1i_1}^*\right)\right] \otimes ... \otimes \left[T\left(a_{ki_k}^*\right)\right]$$

Proof. Proof is straight forward and left to reader (use equation 2.8). □

Remark :

1. $\forall\, T_1 \in \mathcal{L}\left(V_1^*\right) \forall\, T_2 \in \mathcal{L}\left(V_2^*\right) \ldots \forall\, T_k \in \mathcal{L}\left(V_k^*\right)$ $[T_1 \otimes T_2 \otimes \ldots \otimes T_k]$ is multilinear. $\forall\, x, y \in V_1 \otimes V_2 \otimes \ldots \otimes V_k$ since $A_1 \otimes A_2 \otimes \ldots \otimes A_k$ is a basis of $V_1 \otimes V_2 \otimes \ldots \otimes V_k$ there exist unique $\alpha_{i_1,i_2,\ldots,i_k}, \beta_{i_1,i_2,\ldots,i_k} \in \mathbb{C}$ such that,

$$x = \sum_{i_1=1}^{n_1} \ldots \sum_{i_k=1}^{n_k} \alpha_{i_1,\ldots,i_k} a_{1i_1}^* \otimes \ldots \otimes a_{ki_k}^* \qquad y = \sum_{i_1=1}^{n_1} \ldots \sum_{i_k=1}^{n_k} \beta_{i_1,\ldots,i_k} a_{1i_1}^* \otimes \ldots \otimes a_{ki_k}^*$$

$\forall\, \gamma \in \mathbb{C}$,

$$\begin{aligned}[T_1 \otimes \ldots \otimes T_k](x + \gamma y) &= [T_1 \otimes \ldots \otimes T_k]\left(\sum_{i_1=1}^{n_1} \ldots \sum_{i_k=1}^{n_k} (\alpha_{i_1,\ldots,i_k} + \gamma \beta_{j_1,\ldots,j_k})\, a_{1i_1}^* \otimes \ldots \otimes a_{ki_k}^*\right) \\ &= \sum_{i_1=1}^{n_1} \ldots \sum_{i_k=1}^{n_k} (\alpha_{i_1,\ldots,i_k} + \gamma \beta_{i_1,\ldots,i_k}) \left[T_1\left(a_{1i_1}^*\right)\right] \otimes \ldots \otimes \left[T\left(a_{ki_k}^*\right)\right] \\ &= [T_1 \otimes \ldots \otimes T_k](x) + \gamma\, [T_1 \otimes \ldots \otimes T_k](y)\end{aligned}$$

2. $\forall\, T_1 \in \mathcal{L}\left(V_1^*\right) \ldots \forall\, T_i, \tilde{T}_i \in \mathcal{L}\left(V_i^*\right) \ldots \forall\, T_k \in \mathcal{L}\left(V_k^*\right) \forall\, \alpha \in \mathbb{C}$,

$$T_1 \otimes \ldots \otimes \left[T_i + \alpha \tilde{T}_i\right] \otimes \ldots \otimes T_k = [T_1 \otimes \ldots \otimes T_i \otimes \ldots \otimes T_k] + \alpha \left[T_1 \otimes \ldots \otimes \tilde{T}_i \otimes \ldots \otimes T_k\right]$$

This property is straight forward to verify left to reader.

Lemma 2.3.12. $\forall\, T_1 \in \mathcal{L}\left(V_1^*\right) \forall\, T_2 \in \mathcal{L}\left(V_2^*\right) \ldots \forall\, T_k \in \mathcal{L}\left(V_k^*\right)$, $[T_1 \otimes \ldots \otimes T_k]$ is well-defined

Proof. Let $B_1 = \{b_{11}, b_{12}, \ldots, b_{1n_1}\}$ be any orthonormal basis of V_1 and $B_1^* = \{b_{11}^*, b_{12}^*, \ldots, b_{1n_1}^*\}$ be the corresponding dual basis, $B_2 = \{b_{21}, b_{22}, \ldots, b_{2n_2}\}$ be any orthonormal basis of V_2 and $B_2^* = \{b_{21}^*, b_{22}^*, \ldots, b_{2n_2}^*\}$ be the corresponding dual basis, ..., $B_k = \{b_{k1}, b_{k2}, \ldots, b_{kn_k}\}$ be any orthonormal basis of V_k and $B_k^* = \{b_{k1}^*, b_{k2}^*, \ldots, b_{kn_k}^*\}$ be the corresponding dual basis. Since $B_1 \otimes B_2 \otimes \ldots \otimes B_k$ is a basis of $V_1 \otimes V_2 \otimes \ldots \otimes V_k$ there exist unique $\beta_{j_1,j_2,\ldots,j_k} \in \mathbb{C}$ such that,

$$x = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \ldots \sum_{j_k=1}^{n_k} \beta_{j_1,j_2,\ldots,j_k} b_{1j_1}^* \otimes b_{2j_2}^* \otimes \ldots \otimes b_{kj_k}^*$$

Applying $x \in V_1 \otimes V_2 \otimes ... \otimes V_k$ expressed in terms of basis B_1, B_2, ..., B_k to the operator $[T \otimes T_2 \otimes ... \otimes T_k]$ we get that,

$$[T_1 \otimes ... \otimes T_k](x) = \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{j_1,...,j_k} \left[T_1\left(b^*_{1j_1}\right)\right] \otimes ... \otimes \left[T_k\left(b^*_{kj_k}\right)\right]$$

To claim $(T \otimes U)$ is well-defined it is enough to show that

$$\sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{j_1,...,j_k} \left[T_1\left(b^*_{1j_1}\right)\right] \otimes ... \otimes \left[T_k\left(b^*_{kj_k}\right)\right] = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} \left[T_1\left(a^*_{1i_1}\right)\right] \otimes ... \otimes \left[T_k\left(a^*_{ki_k}\right)\right]$$

Let ${}^1M \in \mathbb{C}^{n_1 \times n_1}$ be the transformation matrix from basis A_1 to B_1. Theorem 2.1.5 implies that,

$$\begin{bmatrix} a^*_{11} & a^*_{12} & . & . & a^*_{1n_1} \end{bmatrix} = \begin{bmatrix} b^*_{11} & b^*_{12} & . & . & b^*_{1n_1} \end{bmatrix} {}^1\overline{M}$$

$$\implies a^*_{1i_1} = \sum_{j_1=1}^{n_1} {}^1\overline{M}_{j_1 i_1} b^*_{1j_1} \implies T\left(a^*_{1i_1}\right) = \sum_{j_1=1}^{n_k} {}^1\overline{M}_{j_1 i_1} T\left(b^*_{1j_1}\right)$$

$$...,$$

Let ${}^kM \in \mathbb{C}^{n_k \times n_k}$ be the transformation matrix from basis A_k to B_k. Theorem 2.1.5 implies that,

$$\begin{bmatrix} a^*_{k1} & a^*_{k2} & . & . & a^*_{kn_k} \end{bmatrix} = \begin{bmatrix} b^*_{k1} & b^*_{k2} & . & . & b^*_{kn_k} \end{bmatrix} {}^k\overline{M}$$

$$\implies a^*_{ki_k} = \sum_{j_k=1}^{n_k} {}^k\overline{M}_{j_k i_k} b^*_{kj_k} \implies T\left(a^*_{ki_k}\right) = \sum_{j_k=1}^{n_k} {}^k\overline{M}_{j_k i_k} T\left(b^*_{1j_1}\right)$$

Above equations imply that,

$$\begin{aligned} &\sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} \left[T_1\left(a^*_{1i_1}\right)\right] \otimes ... \otimes \left[T_k\left(a^*_{ki_k}\right)\right] \\ &= \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} {}^1\overline{M}_{j_1 i_1} ... {}^k\overline{M}_{j_k i_k} \alpha_{i_1,...,i_k} \left[T_1\left(b^*_{1j_1}\right)\right] \otimes ... \otimes \left[T_k\left(b^*_{kj_k}\right)\right] \end{aligned}$$

$$= \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \left(\sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} {}^1\overline{M}_{j_1 i_1} ... {}^k\overline{M}_{j_k i_k} \right) \left[T_1\left(b^*_{1j_1}\right)\right] \otimes ... \otimes \left[T_k\left(b^*_{kj_k}\right)\right]$$

Note that

$${}^{B_1 \otimes B_2 \otimes ... \otimes B_k} x\left[j_1, ..., j_k\right] = \beta_{j_1,...,j_k} \qquad {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} x\left[i_1, ..., i_k\right] = \alpha_{i_1,...,i_k}$$

From theorem 2.3.6 we get,

$${}^{B_1 \otimes B_2 \otimes ... \otimes B_k} u\left[j_1, j_2, ..., j_k\right] = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^1\overline{M}_{j_1 i_1} {}^2\overline{M}_{j_2 i_2} ... {}^k\overline{M}_{j_k i_k} {}^{A_1 \otimes A_2 \otimes ... \otimes A_k} u\left[i_1, i_2, ..., i_k\right]$$

$$\Longrightarrow \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{j_1,...,j_k} \left[T_1\left(b^*_{1j_1}\right)\right] \otimes ... \otimes \left[T_k\left(b^*_{kj_k}\right)\right] = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} \left[T_1\left(a^*_{1i_1}\right)\right] \otimes ... \otimes \left[T_k\left(a^*_{ki_k}\right)\right]$$

□

we shall find a basis of $\mathcal{L}\left(V_1 \otimes V_2 \otimes ... \otimes V_k\right)$. Note that $\mathcal{L}\left(V_1 \otimes V_2 \otimes ... \otimes V_k\right)$ is also called tensor product space of operators on $V_1 \otimes V_2 \otimes ... \otimes V_k$.

Definition 2.3.7. Let V_1, V_2, ..., V_k be any finite dimensional inner product spaces over field $\mathbb{C}$ where $\dim(V_i) = n_i \; \forall \; i \in \{1, 2, ..., k\}$. Let $A_1 = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be any orthonormal basis of V_1 and $A_1^* = \{a^*_{11}, a^*_{12}, ..., a^*_{1n_1}\}$ be the corresponding dual basis, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be any orthonormal basis of V_2 and $A_2^* = \{a^*_{21}, a^*_{22}, ..., a^*_{2n_2}\}$ be the corresponding dual basis, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be any orthonormal basis of V_k and $A_k^* = \{a^*_{k1}, a^*_{k2}, ..., a^*_{kn_k}\}$ be the corresponding dual basis. Let $B_1 = \{b_{11}, b_{12}, ..., b_{1n_1}\}$ be any orthonormal basis of V_1 and $B_1^* = \{b^*_{11}, b^*_{12}, ..., b^*_{1n_1}\}$ be the corresponding dual basis, $B_2 = \{b_{21}, b_{22}, ..., b_{2n_2}\}$ be any orthonormal basis of V_2 and $B_2^* = \{b^*_{21}, b^*_{22}, ..., b^*_{2n_2}\}$ be the corresponding dual basis, ..., $B_k = \{b_{k1}, b_{k2}, ..., b_{kn_k}\}$ be any orthonormal basis of V_k and $B_k^* = \{b^*_{k1}, b^*_{k2}, ..., b^*_{kn_k}\}$ be the corresponding dual basis. Let $T_1^* = \{{}^1T_{i_1 j_1} \mid 1 \leq i_1, j_1 \leq n_1\}$ where $\forall \; i_1, j_1, l_1 \in \{1, 2, ..., n_1\}$ ${}^1T_{i_1 j_1} \in \mathcal{L}(V_1^*)$ is defined as follows,

$$\begin{aligned} {}^1T_{i_1 j_1}\left(a^*_{1l_1}\right) &= a^*_{1j_1} \qquad \text{if } l_1 = i_1 \\ &= 0 \qquad \text{if } l_1 \neq i_1 \end{aligned}$$

$$...,$$

Let $T_k^* = \{{}^kT_{i_kj_k} \mid 1 \leq i_k, j_k \leq n_k\}$ where $\forall\ i_k, j_k, l_k \in \{1, 2, ..., n_k\}$ ${}^kT_{i_kj_k} \in \mathcal{L}(V_k^*)$ is defined as follows,

$$\begin{aligned} {}^kT_{i_kj_k}\left(a_{kl_k}^*\right) &= a_{kj_k}^* \qquad \text{if } l_k = i_k \\ &= 0 \qquad \text{if } l_k \neq i_k \end{aligned}$$

Define $T_1^* \otimes T_2^* \otimes ... \otimes T_k^* = \{{}^1T_{i_1j_1}^* \otimes {}^2T_{i_2j_2}^* \otimes ... \otimes {}^kT_{i_kj_k}^* \mid 1 \leq i_1, j_1 \leq n_1, 1 \leq i_2, j_2 \leq n_2, ..., 1 \leq i_k, j_k \leq n_k\}$. Note that each ${}^1T_{i_1j_1} \otimes {}^2T_{i_2j_2} \otimes ... \otimes {}^kT_{i_kj_k} \in T_1^* \otimes T_2^* \otimes ... \otimes T_k^*$ is well-defined and a linear operator on $V_1 \otimes V_2 \otimes ... \otimes V_k$.

Theorem 2.3.13. $T_1^* \otimes T_2^* \otimes ... \otimes T_k^*$ forms a basis of $\mathcal{L}(V_1 \otimes V_2 \otimes ... \otimes V_k)$.

Proof. **Span :**
$\forall\ x \in V_1 \otimes V_2 \otimes ... \otimes V_k$ since $A_1 \otimes A_2 \otimes ... \otimes A_k$ is a basis of $V_1 \otimes V_2 \otimes ... \otimes V_k$ there exist unique $\alpha_{i_1,i_2,...,i_k} \in \mathbb{C}$ such that,

$$x = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} \left[a_{1i_1}^* \otimes ... \otimes a_{ki_k}^*\right]$$

$\forall\ W \in \mathcal{L}(V_1 \otimes V_2 \otimes ... \otimes V_k)$ since W is a linear operator we get,

$$W(x) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,...,i_k} W\left(a_{1i_1}^* \otimes ... \otimes a_{ki_k}^*\right)$$

Since W is an operator $\forall\ i_1 \in \{1, 2, ..., n_1\}$ $i_2 \in \{1, 2, ..., n_2\}$ ... $i_k \in \{1, 2, ..., n_k\}$ there exist $\beta_{i_1,...,i_k,j_1,...,j_k} \in \mathbb{C}$ such that,

$$W\left(a_{1i_1}^* \otimes ... \otimes a_{ki_k}^*\right) = \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{i_1,...,i_k,j_1,...,j_k} a_{1j_1}^* \otimes ... \otimes a_{kj_k}^*$$

$$\implies W(x) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{i_1,...,i_k,j_1,...,j_k} \alpha_{i_1,...,i_k} a_{1j_1}^* \otimes ... \otimes a_{kj_k}^*$$

Note that $\forall\ i_1, j_1 \in \{1, 2, ..., n_1\}$ $\forall\ i_2, j_2 \in \{1, 2, ..., n_2\}$ ... $\forall\ i_k, j_k \in \{1, 2, ..., n_k\}$

since ${}^{1}T_{i_1j_1} \otimes {}^{2}T_{i_2j_2} \otimes ... \otimes {}^{k}T_{i_kj_k}$ is multi-linear,

$$\begin{aligned}\left[{}^{1}T_{i_1j_1} \otimes ... \otimes {}^{k}T_{i_kj_k}\right](x) &= \left[{}^{1}T_{i_1j_1} \otimes ... \otimes {}^{k}T_{i_kj_k}\right]\left(\sum_{l_1=1}^{n_1} ... \sum_{l_k=1}^{n_k} \alpha_{l_1,...l_k} a^*_{1l_1} \otimes ... \otimes a^*_{kl_k}\right) \\ &= \sum_{l_1=1}^{n_1} ... \sum_{l_k=1}^{n_k} \alpha_{l_1,...l_k} \left[{}^{1}T_{i_1j_1}\left(a^*_{1l_1}\right)\right] \otimes ... \otimes \left[{}^{k}T_{i_kj_k}\left(a^*_{kl_k}\right)\right] \\ &= \alpha_{i_1,...,i_k} a^*_{1j_1} \otimes ... a^*_{kj_k}\end{aligned}$$

$$\implies W(x) = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{i_1,...,i_k,j_1,...,j_k} \left[{}^{1}T_{i_1j_1} \otimes ... \otimes {}^{k}T_{i_kj_k}\right](x)$$

$$\implies W = \sum_{i_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \beta_{i_1,...,i_k,j_1,...,j_k} \left[{}^{1}T_{i_1j_1} \otimes ... \otimes {}^{k}T_{i_kj_k}\right]$$

$$\implies T_1^* \otimes T_2^* \otimes ... \otimes T_k^* \text{ spans } \mathcal{L}(V_1 \otimes V_2 \otimes ... \otimes V_k)$$

Linear Independence :

$\forall\ i_1, j_1 \in \{1, 2, ..., n_1\}\ \forall\ i_2, j_2 \in \{1, 2, ..., n_2\}\ ...\ \forall\ i_k, j_k \in \{1, 2, ..., n_k\}$. Consider,

$$\sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_k=1}^{n_k} \alpha_{i_1,j_1,...,i_k,j_k} {}^{1}T_{i_1j_1} \otimes ... \otimes {}^{k}T_{i_kj_k} = 0$$

$\forall\ l_1 \in \{1, 2, ..., n_1\}\ \forall\ l_2 \in \{1, 2, ..., n_2\}\ ...\ \forall\ l_k \in \{1, 2, ..., n_k\}$

Applying $a^*_{1l_1} \otimes a^*_{2l_2} \otimes ... \otimes a^*_{kl_k}$ we get,

$$\sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} ... \sum_{i_k=1}^{n_k} \sum_{j_k=1}^{n_k} \alpha_{i_1,j_1,...,i_k,j_k} \left[{}^{1}T_{i_1j_1}\left(a^*_{1l_1}\right)\right] \otimes ... \otimes \left[{}^{k}T_{i_kj_k}\left(a^*_{kl_k}\right)\right] = 0$$

$$\implies \sum_{j_1=1}^{n_1} ... \sum_{j_k=1}^{n_k} \alpha_{l_1,j_1,...,l_k,j_k} \left[a^*_{1j_1} \otimes ... \otimes a^*_{kj_k}\right] = 0$$

Since $A_1 \otimes A_2 \otimes ... \otimes A_k$ is a basis of $V_1 \otimes V_2 \otimes ... \otimes V_k$ we get that,

$$\alpha_{l_1, j_1, ..., l_k, j_k} = 0 \qquad \forall\ j_1 \in \{1, 2, ..., n_1\}\ ...\ \forall\ j_k \in \{1, 2, ..., n_k\}$$

$\Longrightarrow$ $T_1^* \otimes ... \otimes T_k^*$ is a linearly independent set and forms a basis of $\mathcal{L}\left(V_1 \otimes ... \otimes V_k\right)$

$\square$

Corollary 2.3.14.

$$\boxed{\dim\left(\mathcal{L}\left(V_1 \otimes ... \otimes V_k\right)\right) = \left(\dim\left(V_1 \otimes ... \otimes V_k\right)\right)^2}$$

Chapter 3

Appendix

This chapter aims at generalizing the theory of tensor product spaces to arbitrary bases and fields over finite dimensional vector spaces. Observe that the only constraint placed is that the vector spaces under consideration are finite dimensional. In this chapter proofs are not very detailed which we believe the reader can fill up with the intuition acquainted in the previous chapter.

3.1 1-fold tensor product spaces - dual spaces

3.1.1 Linear Functions

Definition 3.1.1. Let V be a vector space over field $\mathbb{F}$. A function $u : V \to \mathbb{F}$ is called linear if $\forall\ x, y \in V\ \forall\ \alpha \in \mathbb{F}$,

$$u\,(x + y) = u\,(x) + u\,(y)$$

$$u\,(\alpha \cdot x) = \alpha \cdot u\,(x)$$

Let set $S = \{u : V \to \mathbb{F} \mid u \text{ is linear}\}$. Define addition and scalar multiplication on the set S as follows, $\forall\ u, v \in S\ \forall\ x \in V\ \forall\ \alpha \in \mathbb{F}$,

$$[u + v]\,(x) = u\,(x) + v\,(x)$$

$$[\alpha \cdot u]\,(x) = \alpha \cdot u\,(x)$$

A linear function $u \in S$ is called a **1-tensor** or a linear map on V. It is easy to verify that S is a vector space over field $\mathbb{F}$ (Using lemma 2.1.1 we get S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to reader). The vectorspace of all 1-tensors is defined to be the 1−fold tensor product space of V denoted by $\mathcal{L}(V \to \mathbb{F})$ or V^*. In addition, 1−fold tensor product space is also called dual space of V.

3.1.2 Existence of 1-tensors

Definition 3.1.2. Let V be a finite dimensional inner product space over field $\mathbb{F}$ with $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be a basis of V. Define $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ $\forall\ i, j \in \{1, 2, ..., n\}$ $a_i^* \in \mathcal{L}(V \to \mathbb{F})$ is defined as follows $\forall\ x \in V$,

$$\boxed{\begin{aligned} a_i^*(a_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}}$$

Lemma 3.1.1. Let V be a finite dimensional inner product space over field $\mathbb{F}$ with $\dim(V) = n$. Let $A = \{a_1, a_2, ..., a_n\}$ be a basis of V. There exist $A^* = \{a_1^*, a_2^*, ..., a_n^*\}$ such that $\forall\ i, j \in \{1, 2, ..., n\}$ $a_i^* \in \mathcal{L}(V \to \mathbb{F})$ and $\forall\ x \in V$,

$$\begin{aligned} a_i^*(a_j) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}$$

Proof. $\forall\ i \in \{1, 2, ..., n\}$ let a_i^* be a row vector in $\mathbb{F}^n$. That is there exist $\alpha_{i1}, \alpha_{i2}, ..., \alpha_{in} \in \mathbb{F}$ such that,

$$a_i^* = \begin{bmatrix} \alpha_{i1} & \alpha_{i2} & . & . & \alpha_{in} \end{bmatrix}$$

Let $B = \{b_1, b_2, ..., b_n\}$ be any another basis of V. $\forall\ x \in V$ since B is a basis of V there exist unique $\beta_j \in \mathbb{F}$ such that,

$$x = \sum_{j=1}^{n} \beta_j b_j \implies {}^{B}x = \begin{bmatrix} \beta_1 & \beta_2 & . & . & \beta_n \end{bmatrix}^T$$

Action of a_i^* on x is defined as

$$a_i^*(x) = \sum_{j=1}^{n} \alpha_{ij}\beta_j = \begin{bmatrix}\alpha_{i1} & . & . & \alpha_{in}\end{bmatrix} \begin{bmatrix}\beta_1 \\ . \\ . \\ \beta_n\end{bmatrix} = \begin{bmatrix}\alpha_{i1} & . & . & \alpha_{in}\end{bmatrix} \cdot {}^{B}x$$

Next we need to find $\alpha_{i1}, \alpha_{i2}, ..., \alpha_{in} \in \mathbb{F}$ such that,

$$\begin{aligned} a_i^*\left(a_j\right) &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}$$

$\forall\ i, j \in \{1, 2, ..., n\}$. Fix i.

$$\begin{aligned} \begin{bmatrix}\alpha_{i1} & . & . & \alpha_{in}\end{bmatrix} \cdot {}^{B}a_j &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}$$

Since B is a basis of V there exist unique $\beta_{ij} \in \mathbb{F}$ such that,

$$a_j = \sum_{i=1}^{n} \beta_{ij} b_i \implies {}^{B}a_j = \begin{bmatrix}\beta_{1j} & . & . & \beta_{nj}\end{bmatrix}^T \implies \begin{bmatrix}\alpha_{i1} & . & . & \alpha_{in}\end{bmatrix} \begin{bmatrix}\beta_{11} & \beta_{12} & . & . & \beta_{1n} \\ \beta_{21} & \beta_{22} & . & . & \beta_{2n} \\ . & . & . & . & . \\ \beta_{n1} & \beta_{n2} & . & . & \beta_{nn}\end{bmatrix} = e_i$$

Let $M = \begin{bmatrix}\beta_{11} & \beta_{12} & . & . & \beta_{1n} \\ \beta_{21} & \beta_{22} & . & . & \beta_{2n} \\ . & . & . & . & . \\ \beta_{n1} & \beta_{n2} & . & . & \beta_{nn}\end{bmatrix}$. Recall from linear algebra that Since A is a basis of V we get that M is non-singular.

$$\implies \begin{bmatrix}\alpha_{i1} & . & . & \alpha_{in}\end{bmatrix} = e_i \cdot M^{-1}$$

This concludes the existence of a_i^*. Linearity of each a_i^* directly follows from the linearity of coordinates of input arguments. Note that each a_i^* is well-defined since we used arbitrary basis B to obtain a_i^*. $\square$

Remark :

1. Observe that M^{-1} is the transformation matrix from basis A to B for the linear function a_i^*.

2. If A is an orthonormal basis and $\mathbb{F} = \mathbb{C}$ then

$$a_i^*(x) = \overline{{}^B a_i} \odot {}^B x$$

   This general theory when constrained to orthonormal basis is same as the theory developed as in section 2.1.2

3.1.3 Basis of 1-fold tensor product spaces

Lemma 3.1.2. $\forall\, x \in V$,

$$x = \sum_{i=1}^{n} a_i^*(x) a_i$$

Proof. $\forall\, x \in V$ there exist unique $\alpha_i \in \mathbb{C}$ such that,

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

$\forall\, j \in \{1, 2, ..., n\}$,

$$a_j^*(x) = \sum_{i=1}^{n} \alpha_i a_j^*(a_i) = \alpha_j$$

□

Theorem 3.1.3. A^* forms a basis of $\mathcal{L}(V \to \mathbb{F})$

Proof. **Span :** $\forall\, x \in V$.

$$x = \sum_{i=1}^{n} a_i^*(x)\, a_i$$

$\forall\, u \in \mathcal{L}(V \to \mathbb{F})$,

$$u(x) = \sum_{i=1}^{n} u(a_i)\, a_i^*(x) \implies u = \sum_{i=1}^{n} u(a_i)\, a_i^*$$

Linear Independence : Consider,

$$\sum_{i=1}^{n} \alpha_i a_i^* = 0$$

$\forall\ j \in \{1, 2, ..., n\}$,

$$\sum_{i=1}^{n} \alpha_i a_i^* (a_i) = \alpha_j = 0$$

□

3.1.4 Basis transformation

Theorem 3.1.4. Let $B = \{b_1, b_2, ..., b_n\}$ be any another basis of V and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$ be the corresponding dual basis. Let $M \in \mathbb{F}^{n\times n}$ be the transformation matrix from basis A to B i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} M$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{F})$,

$$\boxed{{}^{B^*}u = \left(M^{-1}\right)^T \cdot {}^{A^*}u}$$

Proof. Since M is the transformation from basis A to B we get that M^{-1} is the transformation matrix from basis B to A i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix} M^{-1} \implies b_i = \sum_{j=1}^{n} M_{ji}^{-1} a_j$$

$\forall\ u \in \mathcal{L}(V \to \mathbb{F})$,

$$u(b_i) = \sum_{j=1}^{n} M_{ji}^{-1} a_j \implies \begin{bmatrix} u(b_1) \\ u(b_2) \\ . \\ . \\ u(b_n) \end{bmatrix} = \begin{bmatrix} M_{11}^{-1} & M_{21}^{-1} & . & . & M_{n1}^{-1} \\ M_{12}^{-1} & M_{22}^{-1} & . & . & M_{n2}^{-1} \\ . & . & . & . & . \\ M_{1n}^{-1} & M_{2n}^{-1} & . & . & M_{nn}^{-1} \end{bmatrix} \begin{bmatrix} u(a_1) \\ u(a_2) \\ . \\ . \\ u(a_n) \end{bmatrix}$$

$$\implies {}^{B^*}u = \left(M^{-1}\right)^T \cdot {}^{A^*}u$$

□

3.1.5 Invariance of computation of $1-$tensors under basis transformations

Theorem 3.1.5. $\forall\, u \in \mathcal{L}\,(V \to \mathbb{F})\ \forall\, x \in V$,

$$\boxed{u\,(x) = \sum_{r=1}^{n} {}^{A^*}u[r] \cdot {}^{A}x[r] = \left({}^{A^*}u\right)^T \cdot \left({}^{A}x\right)}$$

Proof. Proof is same as in theorem 2.1.6. Here we provide an alternate proof. Let $B = \{b_1, b_2, ..., b_n\}$ be any another basis of V and $B^* = \{b_1^*, b_2^*, ..., b_n^*\}$ be the corresponding dual basis. It is enough to show that

$$\left({}^{A^*}u\right)^T \cdot \left({}^{A}x\right) = \left({}^{B^*}u\right)^T \cdot \left({}^{B}x\right)$$

Let $M \in \mathbb{F}^{n\times n}$ be the transformation matrix from basis A to B. Then,

$$ {}^{B}x = M \cdot {}^{A}x \qquad\qquad {}^{B^*}u = \left(M^{-1}\right)^T \cdot {}^{A^*}u$$

$$\left({}^{B^*}u\right)^T \cdot \left({}^{B}x\right) = \left(\left(M^{-1}\right)^T \cdot {}^{A^*}u\right)^T \cdot \left(M \cdot {}^{A}x\right) = \left({}^{A^*}u\right)^T \cdot {}^{A}x$$

□

3.2 2-fold tensor product spaces

Definition 3.2.1. Let V, W be vector spaces over field $\mathbb{F}$. A function $u : V \times W \to \mathbb{F}$ is called bi-linear if the following holds,

1. $\forall\ x, y \in V\ \forall\ z \in W$,

$$u(x+y, z) = u(x, z) + u(y, z)$$

2. $\forall\ x \in V\ \forall\ y, z \in W$,

$$u(x, y+z) = u(x, y) + u(x, z)$$

3. $\forall\ x \in V\ \forall\ y \in W\ \forall\ \alpha \in \mathbb{F}$,

$$u(\alpha x, y) = \alpha u(x, y) = u(x, \alpha y)$$

Let set $S = \{u : V \times W \to \mathbb{F} \mid u \text{ is bi-linear}\}$. Define addition and multiplication on the set S as follows, $\forall\ u, v \in S\ \forall\ x \in V\ \forall\ y \in W\ \forall\ \alpha \in \mathbb{F}$,

$$[u+v](x, y) = u(x, y) + v(x, y)$$
$$[\alpha u](x, y) = \alpha u(x, y)$$

A bi-linear function $u \in S$ is called a **2-tensor** or a bi-linear map on $V \times W$. It is easy to verify that S is a vector space over field $\mathbb{F}$ (Using lemma 2.2.1 we get S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all 2-tensors is defined to be the 2-fold tensor product space of V and W denoted by $\mathcal{L}(V \times W \to \mathbb{F})$ or $V \otimes W$.

3.2.1 Tensor products on vector spaces V and W

Definition 3.2.2. Let V, W be any two finite dimensional vector spaces over field $\mathbb{F}$ where $dim\,(V) = n$ and $dim\,(W) = m$. $\forall\ u \in \mathcal{L}\,(V \to \mathbb{F})\ \forall\ v \in \mathcal{L}\,(W \to \mathbb{F})$ Define the tensor product of u and v as a function $[u \otimes v] : V \times W \to \mathbb{F}$ as follows $\forall\ x \in V\ \forall\ y \in W$,

$$\boxed{[u \otimes v]\,(x, y) = u\,(x) \cdot v\,(y)}$$

Remark :

1. $\forall\ u, v \in \mathcal{L}\,(V \to \mathbb{F})$ notice that $u \otimes v \neq v \otimes u$ in general.

Lemma 3.2.1. $\forall\ u \in \mathcal{L}\,(V \to \mathbb{F})\ \forall\ v \in \mathcal{L}\,(W \to \mathbb{F})$, $[u \otimes v]$ is bi-linear.

Proof. Proof is same as in lemma 2.2.2 □

Lemma 3.2.2. 1. $\forall\ u, v \in \mathcal{L}\,(V \to \mathbb{F})\ \forall\ w \in \mathcal{L}\,(W \to \mathbb{F})$,

$$[u + v] \otimes w = u \otimes w + v \otimes w$$

2. $\forall\ u \in \mathcal{L}\,(V \to \mathbb{F})\ \forall\ v, w \in \mathcal{L}\,(W \to \mathbb{F})$,

$$u \otimes [v + w] = u \otimes v + u \otimes w$$

3. $\forall\ u \in \mathcal{L}\,(V \to \mathbb{F})\ \forall\ v \in \mathcal{L}\,(W \to \mathbb{F})\ \forall\ \alpha \in \mathbb{F}$,

$$[\alpha u] \otimes v = u \otimes [\alpha v] = \alpha[u \otimes v]$$

Proof. Proof is same as lemma 2.2.3 □

Remark :

1. $u \otimes v = 0 \iff u = 0$ or $v = 0$. It is straight forward to verify and left to reader.

3.2.2 Basis of 2-fold tensor product spaces

Definition 3.2.3. Let $A = \{a_1, a_2, ..., a_n\}$ be a basis of V and $B = \{b_1, b_2, ..., b_m\}$ be a basis of W. Define $A \otimes B = \{a_i^* \otimes b_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ where a_i^* and b_j^* are defined as in section 3.1.2.

Theorem 3.2.3. $A \otimes B$ is a basis for vector space $\mathcal{L}\left(V \times W \to \mathbb{F}\right)$

Proof. **Span :** $\forall\ x \in V$ there exist unique $\alpha_i \in \mathbb{F}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

$\forall\ y \in W$ there exist unique $\beta_j \in \mathbb{F}$ such that

$$y = \sum_{j=1}^{m} \beta_j b_j$$

$\forall\ u \in \mathcal{L}\left(V \times W \to \mathbb{C}\right)$,

$$u\left(x, y\right) = u\left(\sum_{i=1}^{n} \alpha_i a_i, \sum_{j=1}^{m} \beta_j b_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j u\left(a_i, b_j\right)$$

$\forall\ i \in \{1, 2, ..., n\}\ \forall\ j \in \{1, 2, ..., m\}$,

$$\left[a_i^* \otimes b_j^*\right]\left(x, y\right) = a_i^*\left(x\right) \cdot b_j^*\left(y\right) = \alpha_i \beta_j$$

$$\implies u = \sum_{i=1}^{n} \sum_{j=1}^{m} u\left(a_i, b_j\right)\left[a_i^* \otimes b_j^*\right]$$

$$\implies A \otimes B \text{ spans } \mathcal{L}\left(V \times W \to \mathbb{F}\right)$$

Linear Independence : Consider,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} [a_i^* \otimes b_j^*] = 0$$

$\forall\ p \in \{1, 2, ..., n\}\ \forall\ q \in \{1, 2, ..., m\}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}[a_i^* \otimes b_j^*]\,(a_p, b_q) = 0 \implies \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}a_i^*\,(a_p)\,b_j^*\,(b_q) = \alpha_{pq} = 0$$

$\implies A \otimes B$ is a linearly independent set and a basis of $\mathcal{L}\,(V \times W \to \mathbb{F})$

□

3.2.3 Basis transformation

Theorem 3.2.4. Let $C = \{c_1, ..., c_n\}$ be any another basis of V. Let $D = \{d_1, ..., d_m\}$ be any another basis of W. Let $C \otimes D = \{c_i^* \otimes d_j^* \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Theorem 3.2.3 implies that both $A \otimes B$ and $C \otimes D$ form bases of $\mathcal{L}\,(V \times W \to \mathbb{F})$. Let $M \in \mathbb{F}^{n\times n}$ be the transformation matrix from basis A to C i.e,

$$\begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix} M$$

Let $N \in \mathbb{F}^{m\times m}$ be the transformation matrix from basis B to D i.e,

$$\begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & . & . & . & d_m \end{bmatrix} N$$

$\forall\ u \in \mathcal{L}\,(V \times W \to \mathbb{F})$,

$$\boxed{{}^{C\otimes D}u = \left(M^{-1}\right)^T \cdot {}^{A\otimes B}u \cdot N^{-1}}$$

Proof. Since M is the transformation matrix from basis A to C we get that M^{-1} is the transformation matrix from basis C to A i.e,

$$\begin{bmatrix} c_1 & c_2 & . & . & . & c_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & . & . & . & a_n \end{bmatrix} M^{-1} \implies c_p = \sum_{i=1}^{n} M_{ip}^{-1} a_i$$

Since N is the transformation matrix from basis B to D we get that N^{-1} is the transformation matrix from basis D to B i.e,

$$\begin{bmatrix} d_1 & d_2 & . & . & . & d_m \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & . & . & . & b_m \end{bmatrix} N^{-1} \implies d_q = \sum_{j=1}^{m} N_{jq}^{-1} b_j$$

$\forall\ u \in \mathcal{L}\left(V \times w \to \mathbb{F}\right)$,

$$u\left(c_p, d_q\right) = u\left(\sum_{i=1}^{n} M_{ip}^{-1} a_i, \sum_{j=1}^{m} N_{jq}^{-1} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ip}^{-1} N_{jq}^{-1} u\left(a_i, b_j\right)$$

$$\implies u\left(c_p, d_q\right) = \begin{bmatrix} M_{1p}^{-1} & . & . & M_{np}^{-1} \end{bmatrix} \begin{bmatrix} u\left(a_1, b_1\right) & . & . & u\left(a_1, b_m\right) \\ . & . & . & . \\ u\left(a_n, b_1\right) & . & . & u\left(a_n, b_m\right) \end{bmatrix} \begin{bmatrix} N_{1q}^{-1} \\ . \\ . \\ N_{mq}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} u\left(c_1, d_1\right) & .. & u\left(c_1, d_m\right) \\ . & .. & . \\ u\left(c_n, d_1\right) & .. & u\left(c_n, d_m\right) \end{bmatrix} = \begin{bmatrix} M_{11}^{-1} & .. & M_{n1}^{-1} \\ . & .. & . \\ M_{1n}^{-1} & .. & M_{nn}^{-1} \end{bmatrix} \begin{bmatrix} u\left(a_1, b_1\right) & .. & u\left(a_1, b_m\right) \\ . & .. & . \\ u\left(a_n, b_1\right) & .. & u\left(a_n, b_m\right) \end{bmatrix} \begin{bmatrix} N_{11}^{-1} & .. & N_{1m}^{-1} \\ . & .. & . \\ N_{m1}^{-1} & .. & N_{mm}^{-1} \end{bmatrix}$$

$$\implies {}^{C\otimes D}u = \left(M^{-1}\right)^T \cdot {}^{A\otimes B}u \cdot N^{-1}$$

□

3.2.4 Invariance of computation of 2−tensor under basis transformations

Theorem 3.2.5. $\forall\ u \in \mathcal{L}\left(V \times W \to \mathbb{F}\right)\ \forall\ x \in V\ \forall\ y \in W$,

$$\boxed{u\left(x, y\right) = \sum_{r=1}^{n}\sum_{s=1}^{m} {}^{A}x[r] \cdot {}^{A\otimes B}u[r,s] \cdot {}^{B}y[s] = \left({}^{A}x\right)^T \cdot {}^{A\otimes B}u \cdot {}^{B}y}$$

Proof. Proof is same as in theorem 2.2.7. Here we provide an alternate proof. Let $C = \{c_1, c_2, ..., c_n\}$ be any another basis of V and $D = \{d_1, d_2, ..., d_m\}$ be any another basis of W. It is enough to show that

$$\left({}^{A}x\right)^T \cdot {}^{A\otimes B}u \cdot {}^{B}y = \left({}^{C}x\right)^T \cdot {}^{C\otimes D}u \cdot {}^{D}y$$

Let $M \in \mathbb{F}^{n\times n}$ be the transformation matrix from basis A to C. Then,

$${}^{C}x = M \cdot {}^{A}x$$

Let $N \in \mathbb{F}^{m \times m}$ be the transformation matrix from basis B to D. Then,

$$ {}^{D}y = N \cdot {}^{D}y $$

It is also shown that

$$ {}^{C\otimes D}u = \left(M^{-1}\right)^{T} \cdot {}^{A\otimes B}u \cdot N^{-1} $$

$$ \begin{aligned} \left({}^{C}x\right)^{T} \cdot {}^{C\otimes D}u \cdot {}^{D}y &= \left(M \cdot {}^{A}x\right)^{T} \cdot \left(M^{-1}\right)^{T} \cdot {}^{A\otimes B}u \cdot N^{-1} \cdot N \cdot {}^{B}y \\ &= \left({}^{A}x\right)^{T} \cdot M^{T} \cdot \left(M^{-1}\right)^{T} \cdot {}^{A\otimes B}u \cdot N^{-1} \cdot N \cdot {}^{B}y \\ &= \left({}^{A}x\right)^{T} \cdot {}^{A\otimes B}u \cdot {}^{B}y \end{aligned} $$

□

3.3 k-fold tensor product spaces

3.3.1 Multi-linear Functions

Definition 3.3.1. Let V_1, V_2, ..., V_k be vector spaces over field $\mathbb{F}$. A function $u : V_1 \times V_2 \times ... \times V_k \to \mathbb{F}$ is called multi-linear if the following holds,

1. $\forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_i, \tilde{x}_i \in V_i\ ...\ \forall\ x_k \in V_k$ where $1 \leq i \leq k$,

$$u\left(x_1, x_2, ..., x_i + \tilde{x}_i, ..., x_k\right) = u\left(x_1, x_2, ..., x_i, ..., x_k\right) + u\left(x_1, x_2, ..., \tilde{x}_i, ..., x_k\right)$$

2. $\forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_i \in V_i\ ...\ \forall\ x_k \in V_k$ where $1 \leq i \leq k\ \forall\ \alpha \in \mathbb{F}$,

$$u\left(x_1, x_2, ..., \alpha x_i, ..., x_k\right) = \alpha u\left(x_1, x_2, ..., x_k\right)$$

Let set $S = \{u : V_1 \times V_2 \times ... \times V_k \to \mathbb{F} \mid u \text{ is multi-linear}\}$. Define addition and multiplication on the set S as follows $\forall\ u, v \in S\ \forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_k \in V_k$ $\forall\ \alpha \in \mathbb{F}$,

$$[u + v]\left(x_1, x_2, ..., x_k\right) = u\left(x_1, x_2, ..., x_k\right) + v\left(x_1, x_2, ..., x_k\right)$$

$$[\alpha u]\left(x_1, x_2, ..., x_k\right) = \alpha u\left(x_1, x_2, ..., x_k\right)$$

A multi-linear function $u \in S$ is called a $k-$**tensor** or a multi-linear map on $V_1 \times V_2 \times ... \times V_k$. It is easy to verify that S is a vector space over field $\mathbb{F}$ (Using lemma 2.3.1 we get S is closed under addition and scalar multiplication and rest of the axioms of vector space are easy to verify and are left to the reader). The vector space of all $k-$tensors is defined as the tensor product space of V_1, V_2, ..., V_k denoted by $\mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{F}\right)$ or $V_1 \otimes V_2 \otimes ... \otimes V_k$.

3.3.2 Tensor products on vector spaces V_1, V_2, ..., V_k

Definition 3.3.2. Let V_1, V_2, ..., V_k be finite dimensional inner product spaces over field $\mathbb{C}$ where $dim\,(V_i) = n_i \;\forall\; i \in \{1, 2, ..., k\}$. $\forall\; u_1 \in \mathcal{L}\,(V_1 \to \mathbb{F}) \;\forall\; u_2 \in \mathcal{L}\,(V_2 \to \mathbb{F})$ ... $\forall\; u_k \in \mathcal{L}\,(V_k \to \mathbb{F})$ Define the tensor product of u_1, u_2, ..., u_k as a function $[u_1 \otimes u_2 \otimes ... \otimes u_k] : V_1 \times V_2 \times ... \times V_k \to \mathbb{F}$ as follows $\forall\; x_1 \in V_1 \;\forall\; x_2 \in V_2$ ... $\forall\; x_k \in V_k$,

$$[u_1 \otimes u_2 \otimes ... \otimes u_k]\,(x_1, x_2, ..., x_k) = u_1\,(x_1) \cdot u_2\,(x_2) \cdot ... \cdot u_k\,(x_k) = \prod_{i=1}^{k} u_i\,(x_i)$$

Lemma 3.3.1. $\forall\; u_1 \in \mathcal{L}\,(V_1 \to \mathbb{F}) \;\forall\; u_2 \in \mathcal{L}\,(V_2 \to \mathbb{F})$ ... $\forall\; u_k \in \mathcal{L}\,(V_k \to \mathbb{F})$, $[u_1 \otimes u_2 \otimes ... \otimes u_k]$ is multi-linear.

Proof. Proof is same as in lemma 2.3.2 □

Lemma 3.3.2.

1. $\forall\; u_1 \in \mathcal{L}\,(V_1 \to \mathbb{F})$ ... $\forall\; u_i, \tilde{u}_i \in \mathcal{L}\,(V_i \to \mathbb{F})$ ... $\forall\; u_k \in \mathcal{L}\,(V_k \to \mathbb{F})$ where $1 \leq i \leq k$,

$$[u_1 \otimes ... \otimes [u_i + \tilde{u}_i] \otimes ... \otimes u_k] = [u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k] + [u_1 \otimes ... \otimes \tilde{u}_i \otimes ... \otimes u_k]$$

2. $\forall\; u_1 \in \mathcal{L}\,(V_1 \to \mathbb{F})$ ... $\forall\; u_i \in \mathcal{L}\,(V_i \to \mathbb{F})$ ... $\forall\; u_k \in \mathcal{L}\,(V_k \to \mathbb{F})$ where $1 \leq i \leq k \;\forall\; \alpha \in \mathbb{F}$,

$$[u_1 \otimes ... \otimes [\alpha u_i] \otimes ... \otimes u_k] = \alpha[u_1 \otimes ... \otimes u_i \otimes ... \otimes u_k]$$

Proof. Proof is same as in lemma 3.3.2 □

Remark :

1. $u_1 \otimes ... \otimes u_k = 0 \iff$ at least one of $u_i = 0$. It is straight forward to verify and left to reader.

3.3.3 Basis of k-fold tensor product spaces

Definition 3.3.3. Let $A_1 = \{a_{11}, a_{12}, ..., a_{1n_1}\}$ be a basis of V_1, $A_2 = \{a_{21}, a_{22}, ..., a_{2n_2}\}$ be a basis of V_2, ..., $A_k = \{a_{k1}, a_{k2}, ..., a_{kn_k}\}$ be a basis of V_k. Define $A_1 \otimes A_2 \otimes ... \otimes A_k = \{a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k} \mid 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, ..., 1 \leq i_k \leq n_k\}$.

Theorem 3.3.3. $A_1 \otimes A_2 \otimes ... \otimes A_k$ is a basis for vector space $\mathcal{L}\left(V_1 \times V_2 ... \times V_k \to \mathbb{F}\right)$

Proof. **Span :**

$\forall\ x_1 \in V_1$ there exist unique $\alpha_{1i_1} \in \mathbb{F}$ such that,

$$x_1 = \sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}$$

$\forall\ x_2 \in V_2$ there exist unique $\alpha_{2i_2} \in \mathbb{F}$ such that,

$$x_2 = \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}$$

$$...,$$

$\forall\ x_k \in V_k$ there exist unique $\alpha_{ki_k} \in \mathbb{F}$ such that,

$$x_k = \sum_{i_1=1}^{n_k} \alpha_{ki_k} a_{ki_k}$$

$\forall\ u \in \mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{F}\right)$,

$$\begin{aligned} u\left(x_1, x_2, ..., x_k\right) &= u\left(\sum_{i_1=1}^{n_1} \alpha_{1i_1} a_{1i_1}, \sum_{i_2=1}^{n_2} \alpha_{2i_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_3} \alpha_{ki_k} a_{ki_k}\right) \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{1i_1} \alpha_{2i_2} ... \alpha_{ki_k} u\left(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}\right) \end{aligned}$$

$\forall\ i_1 \in \{1, ..., n_1\}\ i_2 \in \{1, ..., n_2\}\ ...\ i_k \in \{1, ..., n_k\}$,

$$\left[a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}\right]\left(x_1, x_2, ..., x_k\right) = a^*_{1i_1}\left(x_1\right) \cdot a^*_{2i_2}\left(x_2\right) \cdot ... \cdot a^*_{ki_k}\left(x_k\right) = \prod_{j=1}^{k} \alpha_{ji_j}$$

$$\implies u(x_1, x_2, ..., x_k) = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) [a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}] (x_1, x_2, ..., x_k)$$

$$\implies u = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) [a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}]$$

$$\implies A_1 \otimes A_2 \otimes ... \otimes A_k \text{ spans } \mathcal{L}(V_1 \times V_2 ... \times V_k \to \mathbb{F})$$

Linear Independence :

Let $\alpha_{i_1,i_2,...,i_k} \in \mathbb{F}\ \forall\ i_1 \in \{1, ..., n_1\}\ i_2 \in \{1, ..., n_2\}\ ...\ i_k \in \{1, ..., n_k\}$. Consider,

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k}[a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}] = 0$$

$\forall\ j_1 \in \{1, ..., n_1\}\ j_2 \in \{1, ..., n_2\}\ ...\ j_k \in \{1, ..., n_k\}$,

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k}[a^*_{1i_1} \otimes a^*_{2i_2} \otimes ... \otimes a^*_{ki_k}] (a_{1j_1}, a_{2j_2}, ..., a_{kj_k}) = 0$$

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} \alpha_{i_1,i_2,...,i_k} a^*_{1i_1}(a_{1j_1}) a^*_{2i_2}(a_{2j_2}) ... a^*_{ki_k}(a_{kj_k}) = \alpha_{j_1,j_2,...,j_k} = 0$$

$\implies A_1 \otimes A_2 \otimes ... \otimes A_k$ is a linearly independent set and a basis of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{F})$

□

3.3.4 Basis transformation

Theorem 3.3.4. Let $B_1 = \{b_{11}, ..., b_{1n_1}\}$ be any another basis of V_1, let $B_2 = \{b_{21}, ..., b_{2n_2}\}$ be any another basis of V_2, ..., let $B_k = \{b_{k1}, ..., b_{kn_k}\}$ be any another basis of V_k. Let $B_1 \otimes B_2 \otimes ... \otimes B_k = \{b^*_{1j_1} \otimes b^*_{2j_2} \otimes ... \otimes b^*_{kj_k} \mid 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2, ..., 1 \leq j_k \leq n_k\}$. Theorem 3.3.3 implies that $A_1 \otimes A_2 \otimes ... \otimes A_k$ and $B_1 \otimes B_2 \otimes ... \otimes B_k$ form bases of $\mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{F})$. Let ${}^iM \in \mathbb{F}^{n_i \times n_i}$ be the transformation matrix from A_i to B_i i.e,

$$\begin{bmatrix} a_{i1} & a_{i2} & . & . & . & a_{in_i} \end{bmatrix} = \begin{bmatrix} b_{i1} & b_{i2} & . & . & . & b_{in_i} \end{bmatrix} {}^iM$$

where $1 \leq i \leq k$. $\forall\ u \in \mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{F})$,

$$\boxed{{}^{B_1 \otimes B_2 \otimes ... \otimes B_k}u[j_1, j_2, ..., j_k] = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{1}M^{-1}_{i_1 j_1} \cdot {}^{2}M^{-1}_{i_2 j_2} \cdot ... \cdot {}^{k}M^{-1}_{i_k j_k}\, {}^{A_1 \otimes A_2 \otimes ... \otimes A_k}u[i_1, i_2, ..., i_k]}$$

where $1 \leq j_1 \leq n_1 1 \leq j_2 \leq n_2 ... 1 \leq j_k \leq n_k$.

Proof. Since ${}^{1}M$ is the transformation matrix from basis A_1 to B_1 we get that ${}^{1}M^{-1}$ is the transformation from B_1 to A_1 i.e,

$$\begin{bmatrix} b_{11} & b_{12} & . & . & . & b_{1n_1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n_1} \end{bmatrix} {}^{1}M^{-1} \implies b_{1j_1} = \sum_{i_1=1}^{n_1} {}^{1}M^{-1}_{i_1 j_1} a_{1i_1}$$

Since ${}^{2}M$ is the transformation matrix from basis A_2 to B_2 we get that ${}^{2}M^{-1}$ is the transformation from B_2 to A_2 i.e,

$$\begin{bmatrix} b_{21} & b_{22} & . & . & . & b_{2n_2} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & . & . & . & a_{2n_2} \end{bmatrix} {}^{2}M^{-1} \implies b_{2j_2} = \sum_{i_2=1}^{n_2} {}^{2}M^{-1}_{i_2 j_2} a_{2i_2}$$

$$...,$$

Since ${}^{k}M$ is the transformation matrix from basis A_k to B_k, we get that ${}^{k}M^{-1}$ is the transformation from B_k to A_k i.e,

$$\begin{bmatrix} b_{k1} & b_{k2} & . & . & . & b_{kn_k} \end{bmatrix} = \begin{bmatrix} a_{k1} & a_{k2} & . & . & . & a_{kn_k} \end{bmatrix} {}^{k}M^{-1} \implies b_{kj_k} = \sum_{i_k=1}^{n_1} {}^{1}M^{-1}_{i_k j_k} a_{ki_k}$$

$\forall\ u \in \mathcal{L}(V_1 \times V_2 \times ... \times V_k \to \mathbb{F})$,

$$\begin{aligned} u(b_{1j_1}, b_{2j_2}, ..., b_{kj_k}) &= u\left(\sum_{i_1=1}^{n_1} {}^{1}M^{-1}_{i_1 j_1} a_{1i_1}, \sum_{i_2=1}^{n_2} {}^{2}M^{-1}_{i_1 j_2} a_{2i_2}, ..., \sum_{i_k=1}^{n_k} {}^{k}M^{-1}_{i_k j_k} a_{ki_k}\right) \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{1}M^{-1}_{i_1 j_1} {}^{2}M^{-1}_{i_2 j_2} ... {}^{k}M^{-1}_{i_k j_k} u(a_{1i_1}, a_{2i_2}, ..., a_{ki_k}) \end{aligned}$$

$$\implies {}^{B_1 \otimes B_2 \otimes ... \otimes B_k}u[j_1, j_2, ..., j_k] = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} ... \sum_{i_k=1}^{n_k} {}^{1}M^{-1}_{i_1 j_1} {}^{2}M^{-1}_{i_2 j_2} ... {}^{k}M^{-1}_{i_k j_k}\, {}^{A_1 \otimes A_2 \otimes ... \otimes A_k}u[i_1, i_2, ..., i_k]$$

□

3.3.5 Invariance of computation of k-tensors under basis transformations

Theorem 3.3.5. $\forall\ u \in \mathcal{L}\left(V_1 \times V_2 \times ... \times V_k \to \mathbb{F}\right)\ \forall\ x_1 \in V_1\ \forall\ x_2 \in V_2\ ...\ \forall\ x_k \in V_k$,

$$\boxed{u\left(x_1, x_2, ..., x_k\right) = \sum_{i_1=1}^{n_1}\sum_{i_2=2}^{n_2}...\sum_{i_k=1}^{n_k} {}^{A_1\otimes A_2\otimes ...\otimes A_k}u[i_1, i_2, ..., i_k]\,{}^{A}x_1[i_1]\,{}^{A}x_2[i_2]...{}^{A}x_k[i_k]}$$

Proof. Proof is same as in 2.3.7. Here we provide an alternate proof. It is enough to show that

$$\sum_{i_1=1}^{n_1}...\sum_{i_1=k}^{n_k} {}^{A_1\otimes ...\otimes A_k}u[i_1, ..., i_k]\,{}^{A}x_1[i_1]...{}^{A}x_k[i_k] = \sum_{j_1=1}^{n_1}...\sum_{j_k=1}^{n_k} {}^{B_1\otimes ...\otimes B_k}u[j_1, ..., j_k]\,{}^{B_1}x_1[j_1]...{}^{B_k}x_k[j_k]$$

Using the previous theorem we have,

$$\begin{aligned}
&\sum_{j_1=1}^{n_1}...\sum_{j_k=1}^{n_k} {}^{B_1\otimes ...\otimes B_k}u[j_1, ..., j_k]\,{}^{B_1}x_1[j_1]...{}^{B_k}x_k[j_k] \\
&= \sum_{j_1=1}^{n_1}...\sum_{j_k=1}^{n_k}\sum_{i_1=1}^{n_1}...\sum_{i_k=1}^{n_k} {}^{1}M^{-1}_{i_1 j_1}...{}^{k}M^{-1}_{i_k j_k}\,u\left(a_{1i_1}, ..., a_{ki_k}\right)\,{}^{B_1}x_1[j_1]...{}^{B_k}x_k[j_k] \\
&= \sum_{i_1=1}^{n_1}...\sum_{i_k=1}^{n_k} u\left(a_{1i_1}, ..., a_{ki_k}\right)\left(\sum_{j_1=1}^{n_1} {}^{1}M^{-1}_{i_1 j_1}\,{}^{B_1}x_1[j_1]\right)...\left(\sum_{j_k=1}^{n_k} {}^{k}M^{-1}_{i_k j_k}\,{}^{B_k}x_k[j_k]\right)
\end{aligned}$$

$\forall\ j \in \{1, 2, ..., k\}$,

$${}^{B_j}x_j = {}^{j}M \cdot {}^{A_j}x_j \implies {}^{j}M^{-1} \cdot {}^{B_j}x_j = {}^{A_j}x_j$$

$$= \sum_{i_1=1}^{n_1}...\sum_{i_k=1}^{n_k} u\left(a_{1i_1}, ..., a_{ki_k}\right)\,{}^{A_1}x_1[i_1]...{}^{A_k}x_k[i_k]$$

□

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