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I. Hilbert Spaces & Tensor Products

1 Inner Products

Let V be a vector space. An *inner product* on V is a map $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that:

0) $(v, v) \geq 0$
i) $v \neq 0 \Rightarrow (v, v) > 0$
ii) $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$
iii) $(u, v) = \overline{(v, u)}$

A vector space V together with an inner product is often called an *inner product space*.

If $\{|i\rangle : i \in I\}$ is an orthonormal basis for V, then for $u = \sum_i \alpha_i |i\rangle$, $v = \sum_i \beta_i |i\rangle$, we have the useful formula:

$$(u, v) = \sum_i \alpha_i \overline{\beta_i}$$

.

It's easy to see that $||v|| = (v, v)^{\frac{1}{2}}$ defines a norm on V. A *Hilbert space* is an inner product space which is complete with respect to the norm. By *complete* we mean that every sequence of vectors $\{v_n\}_{n \in \mathbb{N}}$ with the property that:

$$(\forall \epsilon)(\exists N)(k, j \Rightarrow ||v_k - v_j|| < \epsilon),$$

has a limit in V. This last property is significant only for the infinite dimensional case, since it is trivially satisfied in finite dimensions. Hilbert spaces are fundamental in the world of mathematics, because an inner product yields a notion of geometry in the space. For example, in a Hilbert space, there is always a vector that realizes the minimum distance from a point to a subspace. This is not the case in a general normed vector space.

A very useful fact about Hilbert spaces is that all Hilbert spaces of a given dimension are isomorphic.

2 Tensor Products

A single quantum bit is a unit vector in the Hilbert space $\mathbb{C}^2$. Now suppose we have two quantum bits. How do we write them together? We need a new Hilbert space which captures the interaction of the two bits.

If V, W are vector spaces with bases $\{v_1 \ldots v_n\}$, $\{w_1 \ldots w_m\}$, the *tensor product* $V \otimes W$ of V and W is a nm-dimensional vector space which is spanned by elements of the form $v \otimes w$ - called *elementary tensors*. These elementary tensors behave bilinearly, i.e., we have the relations

$$\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$$

$$u \otimes v + w \otimes v = (u + w) \otimes v \qquad u \otimes v + u \otimes w = u \otimes (v + w)$$

A basis for the tensor product space consists of the vectors: $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$, and thus a general element of $V \otimes W$ is of the form

$$\sum_{i,j} \alpha_{ij} v_i \otimes w_j$$

This definition extends analogously to tensor products with more than two terms.

The tensor product space is also a Hilbert space with the inherited inner product:

$$(v \otimes w, v' \otimes w') = (v, v')(w, w')$$

As it turns out, a two bit system is conveniently represented by a unit vector in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. As I remarked above, $\mathbb{C}^2 \otimes \mathbb{C}^2$ is necessarily isomorphic to $\mathbb{C}^4$ since there is only one complex four dimensional Hilbert space, but as we will see, in the world of quantum mechanics it is convenient to be able to "construct" the larger space from the smaller ones.

Using dirac "ket" notation, we write the basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Remark: $|0\rangle \otimes |0\rangle$ will often be written as $|0\rangle|0\rangle$ or $|00\rangle$.

In general, we represent an n-particle system by n copies of $\mathbb{C}^2$ tensored together. This means that an n-particle system is represented by a 2^n dimensional space!

3 Linear Operators

When we represent quantum bits as unit vectors in a Hilbert space, a quantum gate can be represented as a linear transformation or *linear operator* on this space. Not every linear operator is a quantum gate, though. As we will see later in the course, the action of a quantum gate must preserve the inner product of the underlying Hilbert space. An inner product preserving operator is called a *unitary* operator. Such operators are isomorphisms of Hilbert spaces since the inner product provides all the structure.

If U is a linear operator, we use the notation $U^\dagger$ to represent the *adjoint* of U. The adjoint is the unique operator that satisfies

$$(Uv, w) = (v, U^\dagger w)$$

for all vectors v, w in the Hilbert space. When U is represented by a matrix, $U^\dagger$ is represented by the conjugate transpose of this matrix.

It's easy to show that U is *unitary* if and only if

$$U^\dagger = U^{-1}$$

i.e. if

$$UU^\dagger = U^\dagger U = I$$

4 Linear Operators and Quantum Circuits

Now we will see the full power of tensor products as a means of computing the action of a quantum circuit.

The general set-up is this: Suppose we have a multi-bit circuit which sends the first two bits through one quantum gate, and the remaining bits through a second gate. In the language of linear operators, we are simultaneously performing a transformation U on the Hilbert space $\mathbb{C}^4$, and a transformation V on the Hilbert space which describes the remaining bits. How do we compute the net effect of the entire circuit?

Let's look at a specific example:

For simplicity, let's consider a 3-bit circuit. Let U be a control NOT gate, and V the identity transformation. Thus, we are considering the quantum circuit:

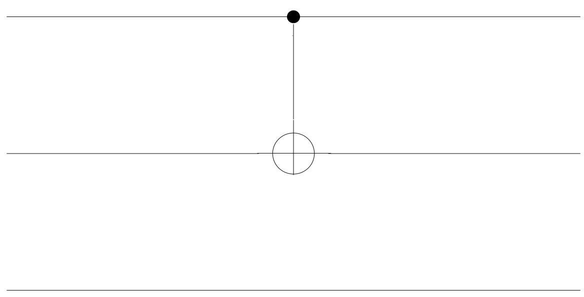

Figure 1: attempt to outdo Ike

When we perform the control NOT operation on the first two bits, we are effecting the following transformation on $\mathbb{C}^4$:

$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to |11\rangle$$
$$|11\rangle \to |10\rangle$$

Using this ordering of the basis vectors, this corresponds to the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

For the final bit, we just have a two by two identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

operating on $\mathbb{C}^2$. The action of this circuit can be conveniently represented by letting our underlying Hilbert space be $\mathbb{C}^4 \otimes \mathbb{C}^2$, and performing the operation U on the first term of the tensor product and V on the second term.

Now, it will sometimes be convenient to think of this quantum circuit as a single operator acting on $\mathbb{C}^8$. If play around for a bit, we see that the appropriate matrix representation of this new operator, $U \otimes V$, is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We obtained this new matrix by replacing each entry in the first matrix by a copy of the second matrix scaled by this entry. This matrix depended, of course, on the ordering I chose for the basis elements of the three spaces $\mathbb{C}^4, \mathbb{C}^2$ and $\mathbb{C}^8$, so let me add that I'm assuming a lexicographical ordering throughout.

In general, if $A = (a_{ij})$ and $B = (b_{ij})$ then $A \otimes B = (a_{ij}B)$. Furthermore, tensoring is an associative operation, so for more than two terms, we can apply the above formula repeatedly.

Thus, we have two equivalent ways of computing the net effect of a quantum circuit. We can either break up the space into a tensor product and apply the gate operators to each piece and tensor the result, or we can tensor the individual gate operators together to get a single operator representing the action of the entire circuit.

5 Summary

In this lecture we have seen how to describe the action of quantum gates on quantum bits by linear operators on Hilbert spaces. When we move from quantum gates to quantum circuits, we see that tensor products enable us to easily go back and forth between two convenient descriptions of the net effect of an entire quantum circuit.